

## FLOW AND POLLUTION TRANSPORT IN THE STREET CANYON

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**Abstract.** Air pollution is one of the serious problems in almost all countries, especially those with high population density and large industrial centers. We develop a mathematical model based on Navier-Stokes equations for viscous incompressible flow and diffusion-convection equation describing pollution transport, and solve the model using finite element method (FEM). A simple algebraic turbulence model is included with the turbulent viscosity scaled for urban area problems. We present the recent numerical results of Navier-Stokes flow and pollution transport in the 2D street canyon.

**Key words.** FEM, atmospheric boundary layer, Navier-Stokes equations.

**AMS subject classifications.** 35K60, 35K65, 65N06, 68U10

**1. Introduction.** We use a 2D model of air flow and pollution transport on a polygonal domain  $\Omega$  which represents a vertical cut through the street area. We consider the case of stationary Navier-Stokes flow and diffusion-convection equation for one type of pollutant. We solve the following system of equations on  $(0, T) \times \Omega$ :

$$\begin{aligned} \frac{\partial c(t, x)}{\partial t} + \vec{v}(x) \operatorname{grad} c(t, x) &= D \Delta c(t, x) + f(t, x), \\ \vec{v}(x) \nabla \vec{v}(x) - \nu \Delta \vec{v}(x) + \operatorname{grad} p(x) &= \vec{g}(x), \\ \operatorname{div} \vec{v}(x) &= 0, \\ c(0, x) &= c_0(x) \quad x \in \Omega, \\ \frac{\partial c}{\partial \vec{n}} \Big|_{terr} &= 0, \\ \vec{v} \Big|_{terr} &= \vec{v}_t, \end{aligned}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain,  $c(t, x)$  is the concentration of the pollutant,  $\vec{v}(x)$  is the velocity,  $c_0$  is the initial condition for concentration,  $\vec{n}$  is the unit outer normal and  $terr$  denotes the terrain. The term  $f(t, x)$  represents the pollution source and  $\vec{g}(x)$  is the external force.

On the surface, we use the Neumann boundary condition for concentration and the Dirichlet boundary condition for velocity, which are appropriate from the physical point of view; see [9]. We use either Dirichlet or Neumann boundary conditions on the other parts of the boundary, like  $c|_{in} = c_i$ ,  $\vec{v}|_{in} = \vec{v}_i$ ,  $\frac{\partial c}{\partial \vec{n}}|_{out} = 0$ ,  $\frac{\partial \vec{v}}{\partial \vec{n}}|_{out} = 0$ .

**2. Weak formulation and numerical solution of the problem.** The Navier-Stokes problem is weakly formulated as follows. Let  $V = (\dot{W}_2^{(1)}(\Omega))^2$ ,  $X = (W_2^{(1)}(\Omega))^2$ ,  $H = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\}$ ,  $\vec{w} \in X$ :  $\vec{w}|_{\partial\Omega} = \vec{v}_\Gamma$  in the weak sense,  $\int_{\partial\Omega} \vec{w} \vec{n} \, dS = \int_{\Omega} \operatorname{div} \vec{u} \, dx = 0$  for Dirichlet boundary condition. We denote  $\vec{u} = \vec{v} - \vec{w}$

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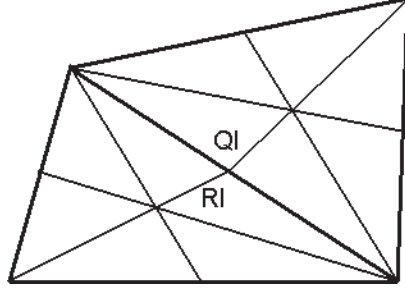


FIG. 2.1. Lumped regions

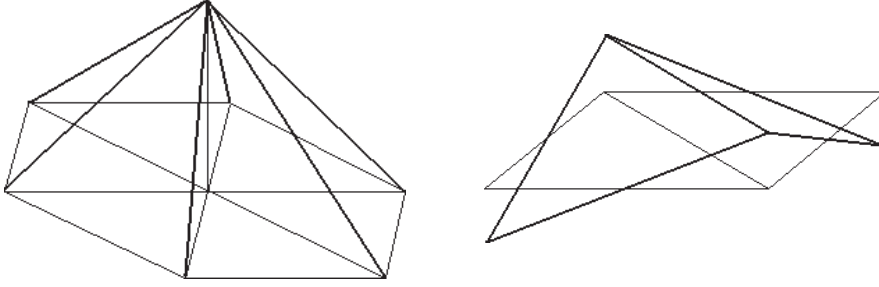


FIG. 2.2. Lagrange and Cruzeix-Raviart elements

$$((\vec{w}, \vec{s})) = \int_{\Omega} \sum_{i,j=1}^2 \frac{\partial w_i}{\partial x_j} \frac{\partial s_i}{\partial x_j} = (\nabla \vec{w}, \nabla \vec{s}), \quad b(\vec{u}, \vec{v}, \vec{s}) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^2 (u_j \frac{\partial v_i}{\partial x_j} s_i - u_j v_i \frac{\partial s_i}{\partial x_j}).$$

We seek  $v \in X$  and  $p \in H$ , such that:

$$\begin{aligned} ((\vec{v}, \vec{s})) + b(\vec{v}, \vec{v}, \vec{s}) - (p, \operatorname{div} \vec{s}) &= (\vec{g}, \vec{s}) - ((\vec{w}, \vec{s})) \quad \forall \vec{s} \in V, \\ (q, \operatorname{div} \vec{u}) &= -(q, \operatorname{div} \vec{w}) \quad \forall q \in H. \end{aligned}$$

The index  $h$  denotes finite-dimensional subspaces  $V^h \subset V$ ,  $X^h \subset X$ ,  $H^h \subset H$ . The mixed formulation in finite-dimensional case stands:

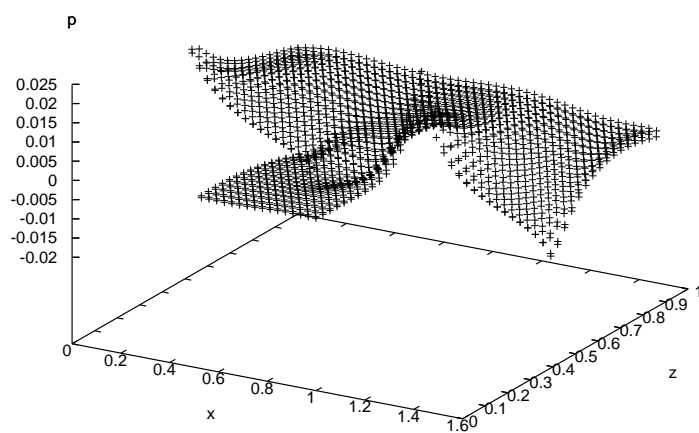
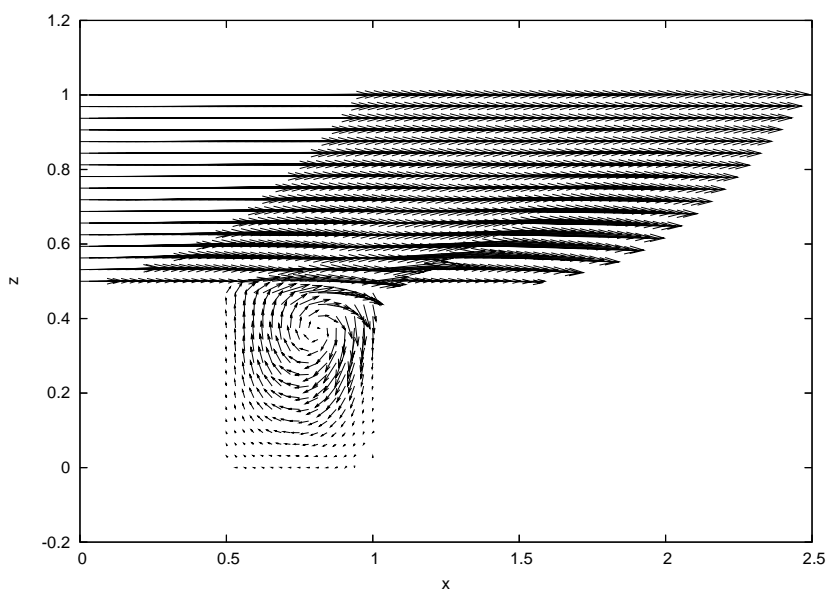
$$\begin{aligned} ((\vec{v}^h, \vec{s}))_h + b_h(\vec{v}^h, \vec{v}^h, \vec{s}^h) - (p^h, \operatorname{div}_h \vec{s}^h) &= (\vec{g}^h, \vec{s}^h)_h - ((\vec{w}^h, \vec{s}^h))_h, \\ (q, \operatorname{div}_h \vec{u}^h)_h &= -(q, \operatorname{div}_h \vec{w}^h)_h \quad \forall \vec{s} \in V^h \quad \forall q \in H^h. \end{aligned}$$

The nonlinear term  $b_h(\vec{u}^h, \vec{u}^h, \vec{s}^h)$  must be computed iteratively. Direct application of this approach results in oscillations in the solution. We use the unwinding technique proposed by [6], based on dual triangulation. This approach eventually leads to the iterative scheme:

$$(2.1) \quad \begin{pmatrix} \mathbf{A}(u^k) & \mathbf{B} \\ \mathbf{B}^T & 0 \end{pmatrix} \begin{pmatrix} u^k \\ -p^k \end{pmatrix} = \begin{pmatrix} G(u^k) \\ H \end{pmatrix}.$$

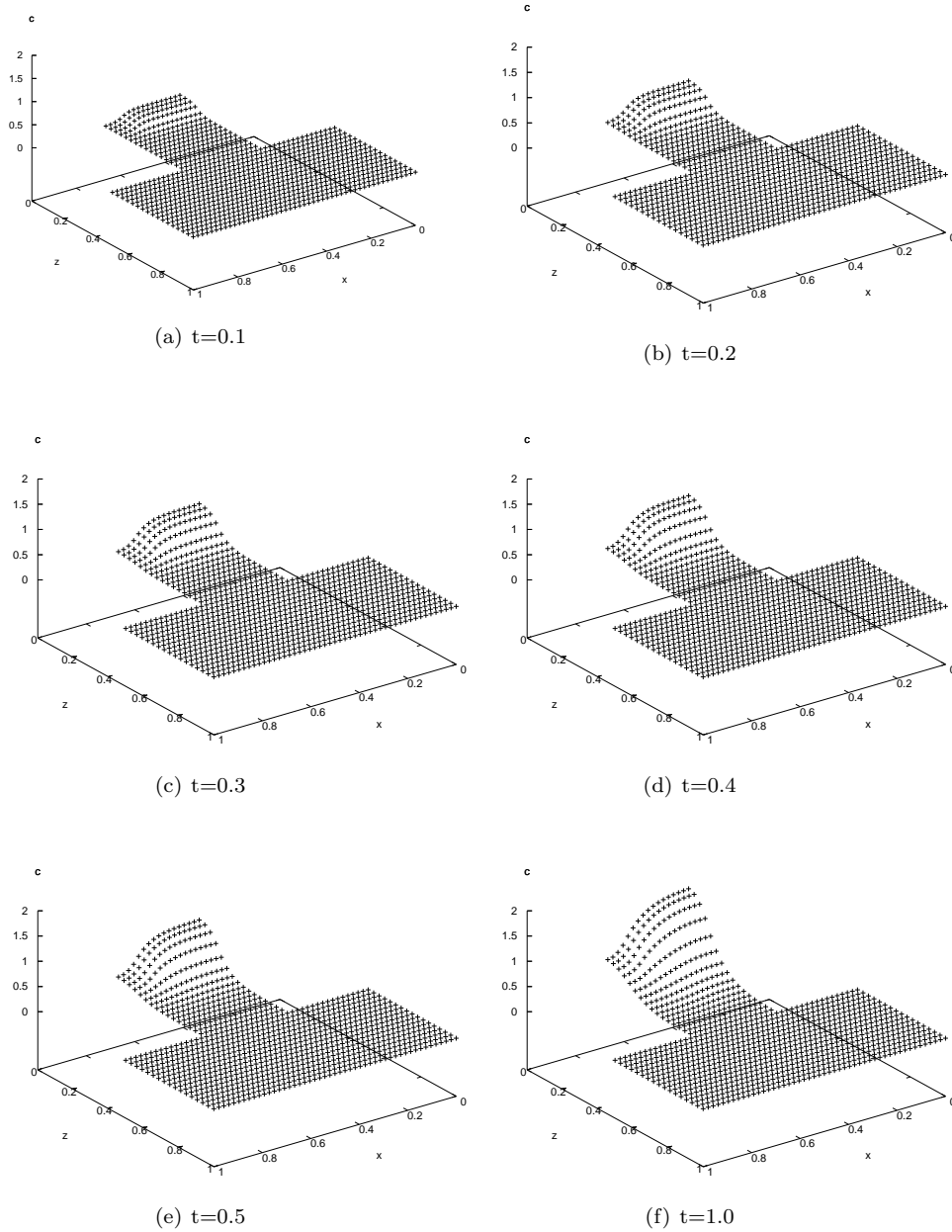
For the Diffusion-Convection Equation, we use implicit Rothe method [4] and the method of characteristics [5] which separates diffusion from convection. This approach uses pre-computed velocity field, and is therefore independent of the flow type used. Application of these methods leads to the following linear system with a positive-definite matrix; see [10]. For  $i = 1, \dots, m$ :

$$\sum_{j=1}^m \alpha_j^k [(v_j, v_i) + \tau D(\nabla v_j, \nabla v_i)] = (\tau f^k + c^{k-1} \circ \varphi^k, v_i) - \tau D(\nabla w, \nabla v_i).$$

FIG. 3.1. *Navier-Stokes flow - pressure*FIG. 3.2. *Navier-Stokes flow - velocity*

**Numerical Solution using the Finite Element Method.** We have chosen the finite element method in order to treat different terrain shapes easily. We use the linear Lagrange elements for concentration, Cruzeix-Raviart (Fig. 2.2) elements for velocity and piecewise constant elements for pressure.

Current mesh structure allows storing of multiple meshes together with their hierarchic structure to support the efficient multigrid solver which is being developed. We can treat two different types of boundary conditions on each part of the boundary.

FIG. 3.3. *Time evolution of concentration*

**3. Results.** We show the recent results of steady state Navier-Stokes flow in the street canyon with Reynolds number  $Re = 1000$  using parabolic velocity profile on the inlet. Note that the pressure is determined up to a constant; we have chosen zero as a mean value in this case. The domain  $\Omega$  has unit size, and the parameters are:  $\vec{g} = 0$ ,  $D = 10^{-2}$ . A constant source of pollution is located at the street level simulating

exhaust gases from the traffic. We can observe the cumulation of pollutant at the leeward side of the the street. This is in agreement with the experimental results, and occurs due to the circular flow in the canyon.

Due to relatively low Reynolds number and isothermal model, only a small amount of pollutant escapes the canyon because of convection.

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