Mathematical Modelling Group

Numerical Analysis of Multiphase Porous Media Flow in Groundwater Contamination Problems

Radek Fučík[1]

Exact Solutions of Two-Phase Flow in Porous Media

DSEM 4/6/2006

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Groundwater protection

- Leaking tanks
- Disposal of industrial waste products
- Agricultural activity (fertilization)
- Groundwater depletion
- Salt water intrusion

Forecasting problems

- Oil and natural gas extraction
- Transport of contaminations in subsurface

Model restrictions

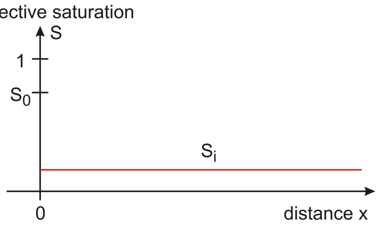
- Two-phase flow system
- Incompressible fluids
- Immiscible phases
- One-dimensional benchmark solutions

Purpose of exact solutions

- Complex numerical schemes validation
- Understanding immiscible two-phase flow

1D two-phase flow equation (advection, diffusion)

$$\vartheta\Phi\frac{\partial S}{\partial t} + ARt^{-\frac{1}{2}}\frac{\partial f(S)}{\partial x} - \frac{\partial}{\partial x}\left(D(S)\,\frac{\partial S}{\partial x}\right) = 0$$



initial state at t = 0 s

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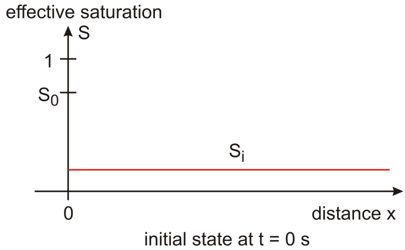
- $S = S(t,x) \in [0,1]$ ... **effective saturation** - unknown function



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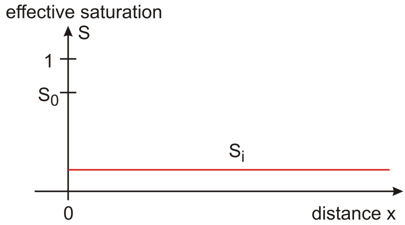

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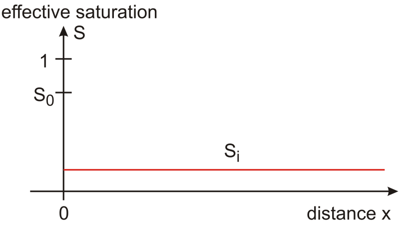

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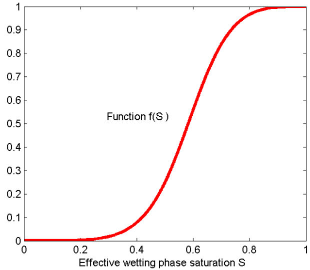

Exact Solution

McWhorter and Sunada Problem

1D two-phase flow equation (advection, diffusion)

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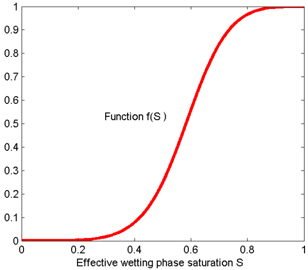

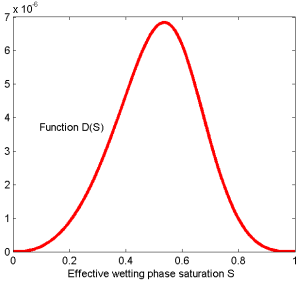

Exact solution
Integral equation

- Exact solution is implicitly obtained from

$$x = F'(S)\frac{2A(1 - Rf(S_i))}{\vartheta\Phi}\sqrt{t}$$

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- Function $F(S)$ has to be computed numerically from

$$F(S) = 1 - \frac{\int\limits_{S}^{S_0} \frac{(v-S)\ D(v)}{F(v)-\varphi(v)}\,\mathrm{d}v}{\int\limits_{S_i}^{S_0} \frac{(v-S_i)\ D(v)}{F(v)-\varphi(v)}\,\mathrm{d}v}.$$

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(the exact solution is therefore quasi-analytical)

- Exact solution is implicitly obtained from

$$x = F'(S)\frac{2A(1 - Rf(S_i))}{\vartheta\Phi}\sqrt{t}$$

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$$F_{k+1}(S) = 1 - \frac{\int\limits_{S}^{S_0} \frac{(v-S)\ D(v)}{F_k(v)-\varphi(v)}\,\mathrm{d}v}{\int\limits_{S_i}^{S_0} \frac{(v-S_i)\ D(v)}{F_k(v)-\varphi(v)}\,\mathrm{d}v}.$$

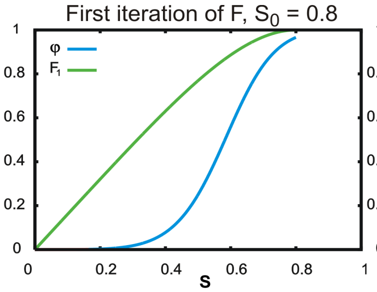
(the exact solution is therefore quasi-analytical)

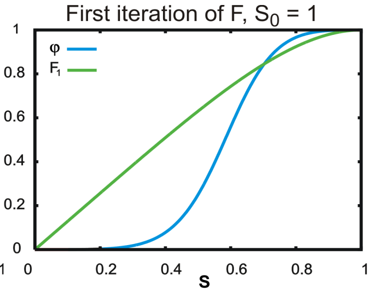
- McWhorter and Sunada suggest to solve the integral equation by iterations with $F_0 \equiv 1$.

1. The suggested iterative scheme fails for $R = 1$ and $S_0 > S_{0\ crit}$ due to singularity in the first iteration

First iteration F_1 can be integrated analytically (with $k = 0$) from

$$F_{k+1}(S) = 1 - \frac{\displaystyle\int_S^{S_0} \frac{(s-S)\ D(s)}{F_k-\varphi(s)}\,\mathrm{d}s}{\displaystyle\int_{S_i}^{S_0} \frac{s\ D(s)}{F_k-\varphi(s)}\,\mathrm{d}s}.$$





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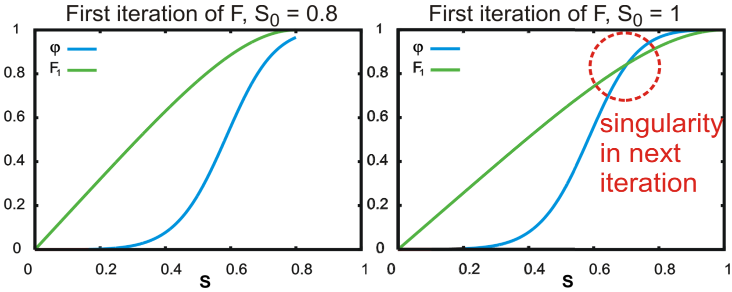

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2. Is there another way to obtain F ?

Substitution $G \equiv \frac{D}{F-\varphi}$ allows to obtain modified integral equations :

variant A :

$$G_{k+1}(S) = D(S) + G_k(S)\left(\varphi(S) + \frac{\int_S^{S_0}(v - S_e)\, G_k(v)\,\mathrm{d}v}{\int_{S_i}^{S_0}(v - S_i)\, G_k(v)\,\mathrm{d}v}\right)$$

variant B :

$$G_{k+1}(S) = \left(D(S) + G_k(S)\,\varphi(S)\right)\left(1 - \frac{\int_{S_e}^{S_0}(v - S)\, G_k(v)\,\mathrm{d}v}{\int_{S_i}^{S_0}(v - S_i)\, G_k(v)\,\mathrm{d}v}\right)^{-1}$$

Modified integral equation is convergent for all $R \in [0, 1]$ and $S_0 \in (S_i, 1 - \epsilon)$, Modified integral equation is convergent for all $R \in [0, 1]$ and $S_0 \in (S_i, 1 - \epsilon)$, but for $S_0 \to 1$:

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- number of iterations increases considerably
- iterative process fails due to computer errors

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Troubles and Questions

Overview

Mathematical Modelling Group

1. The suggested iterative scheme fails for $R = 1$ and $S_0 > S_{0\ crit}$ due to singularity in the first iteration !
2. Is there another way to obtain F ? Yes.
3. Is it possible to generalize the range of the input parameters ?

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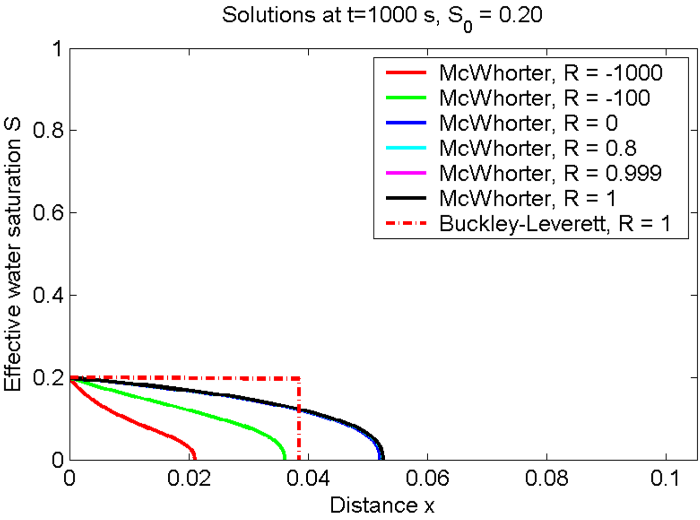
4. Is the Buckley-Leverett solution the limiting solution as $S_0 \to 1$?

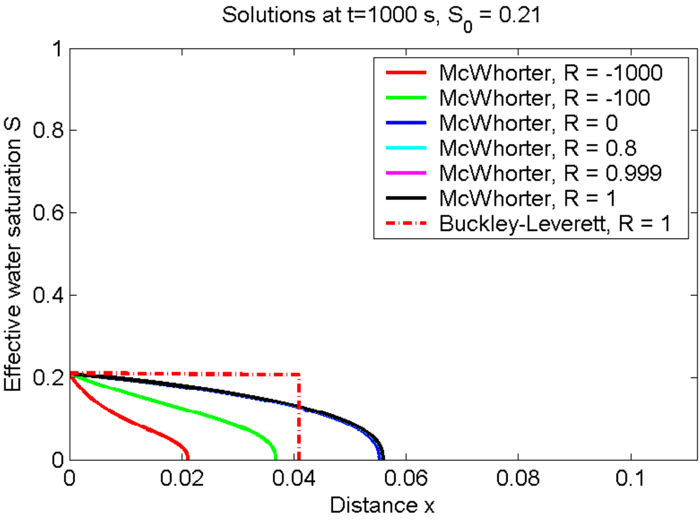
The Buckley-Leverett solution is the analytical solution (MOC) of the hyperbolic problem

$$\vartheta\Phi\frac{\partial S}{\partial t} + ARt^{-\frac{1}{2}}\frac{\partial f(S)}{\partial x} = 0$$

i.e. the two-phase flow equation without the diffusion term



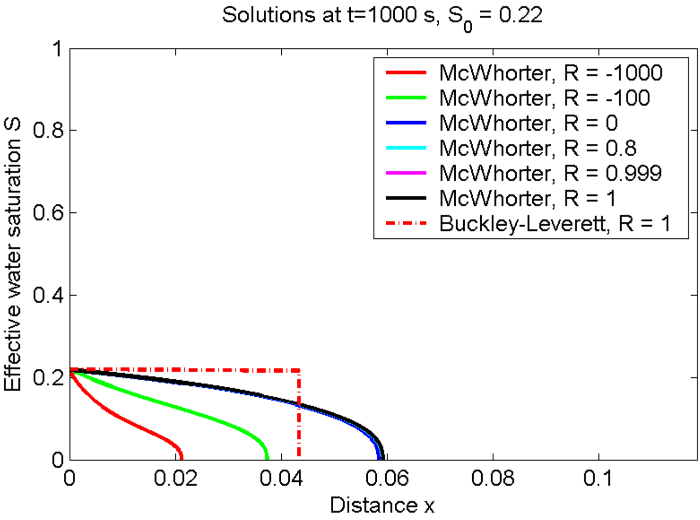
Solutions at t=1000 s, S0 = 0.20
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.02
0.04
0.06
0.08
0.1
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S0 = 0.21
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0.02
0.04
0.06
0.08
0.1

Troubles and Questions

$S_0 \longrightarrow 1$ Demonstration



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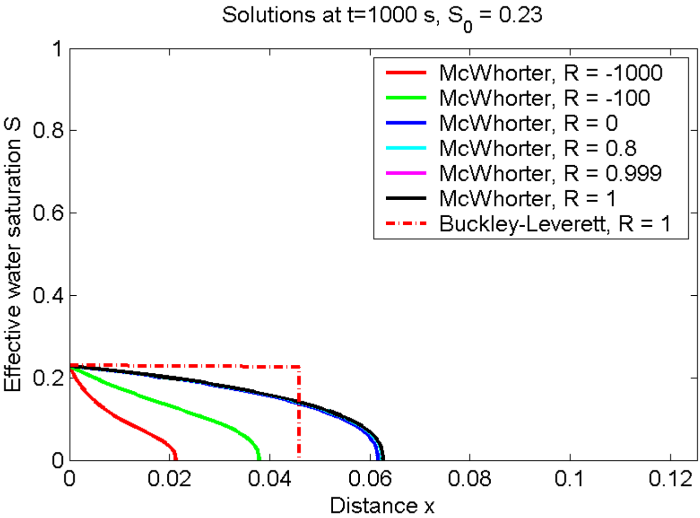
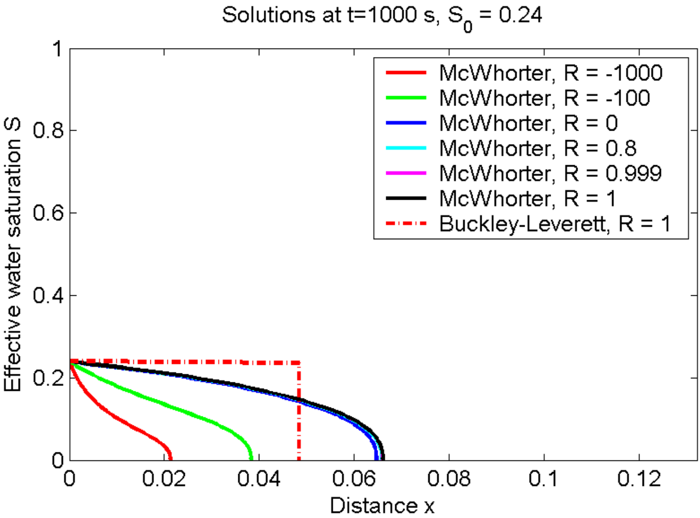
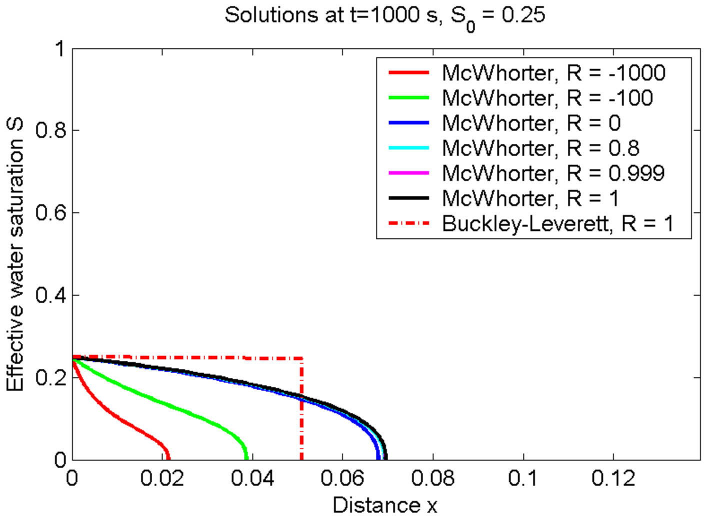
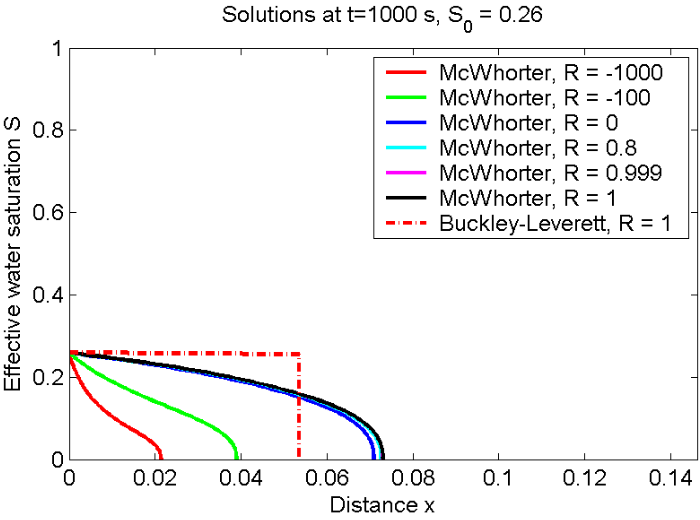



Solutions at t=1000 s, S0 = 0.24
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.02
0.04
0.06
0.08
0.1
0.12
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1


Solutions at t=1000 s, S0 = 0.25
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.02
0.04
0.06
0.08
0.1
0.12
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

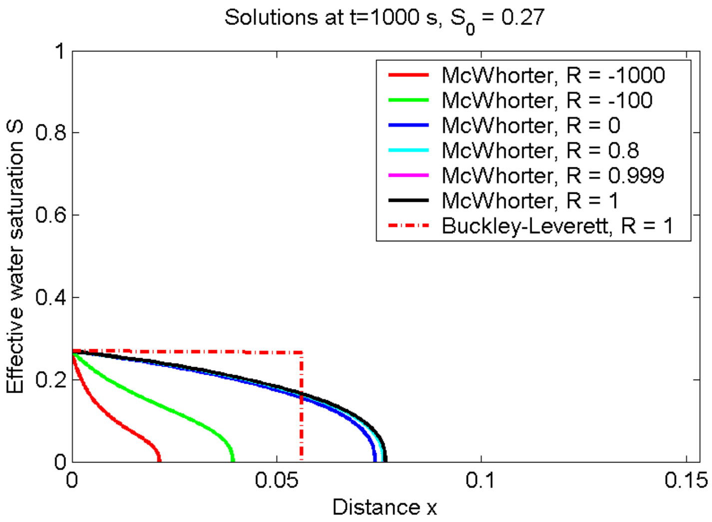
Troubles and Questions

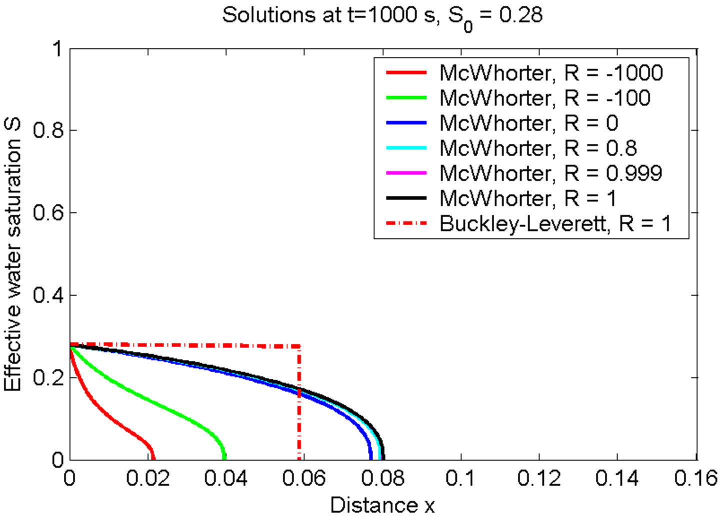
$S_0 \longrightarrow 1$ Demonstration



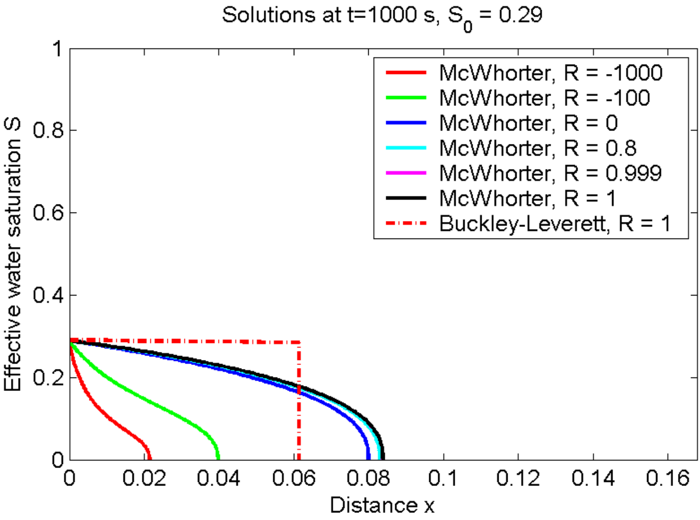
Troubles and Questions

$S_0 \to 1$ Demonstration

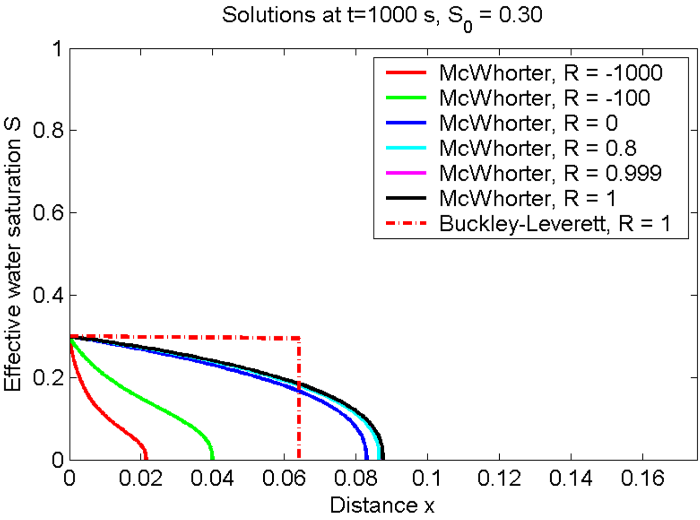




Solutions at t=1000 s, S_0 = 0.28
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16



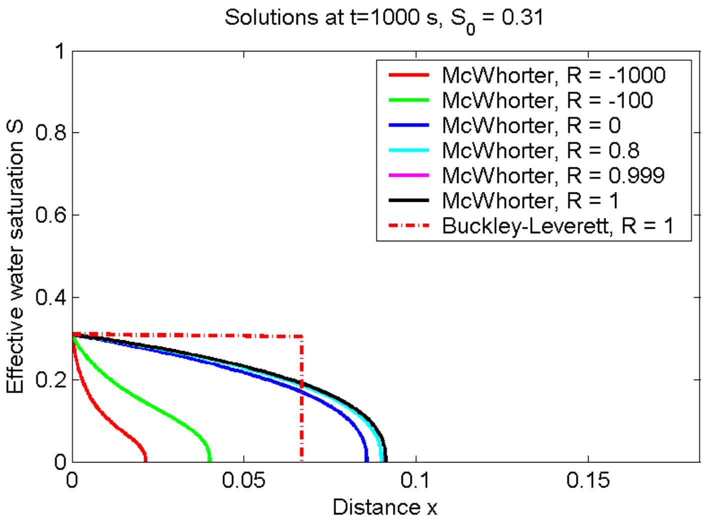
Solutions at t=1000 s, S0 = 0.29
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

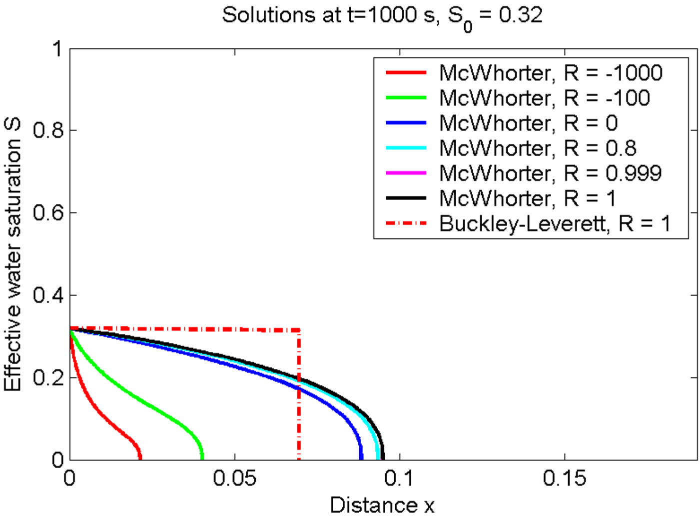


Solutions at t=1000 s, S_0 = 0.30
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16

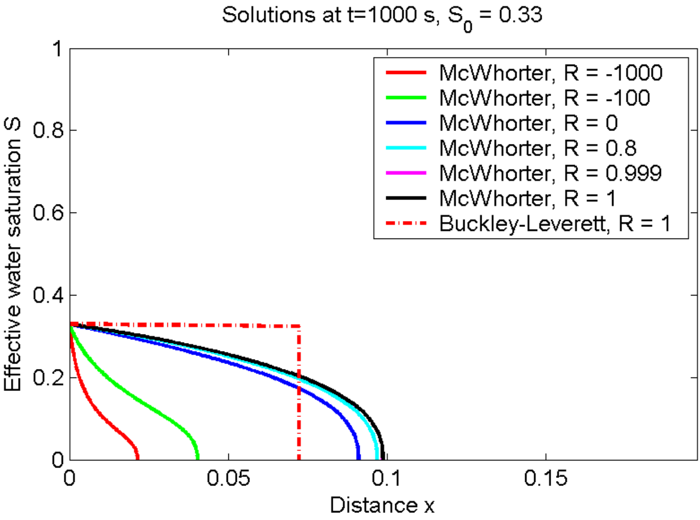
Troubles and Questions

$S_0 \longrightarrow 1$ Demonstration

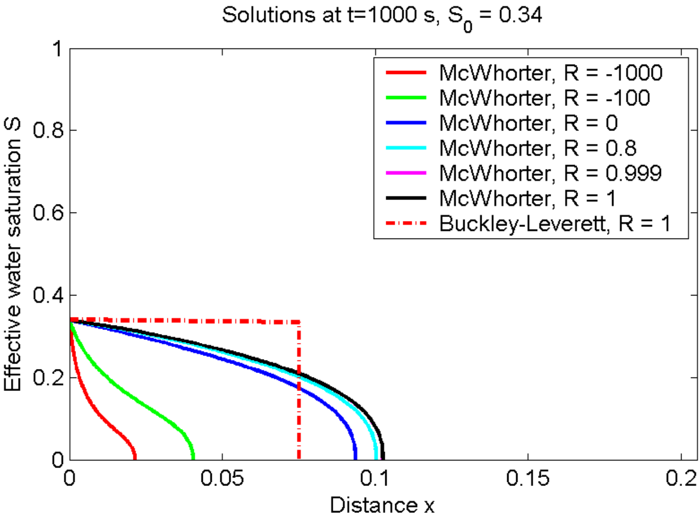




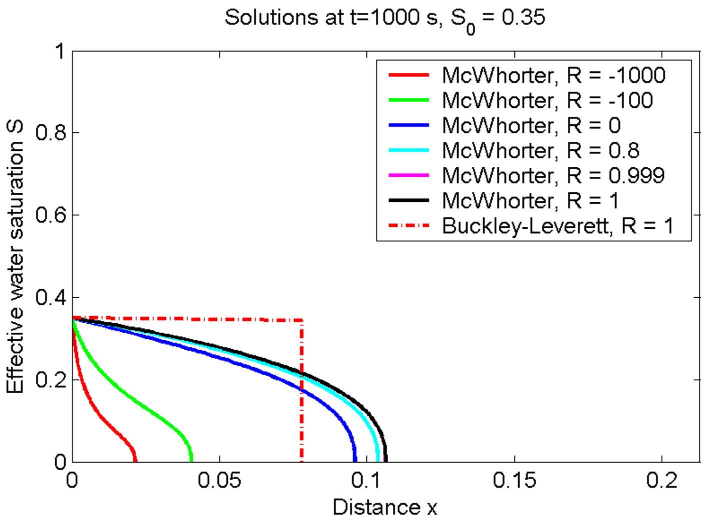
Solutions at t=1000 s, S0 = 0.32
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15



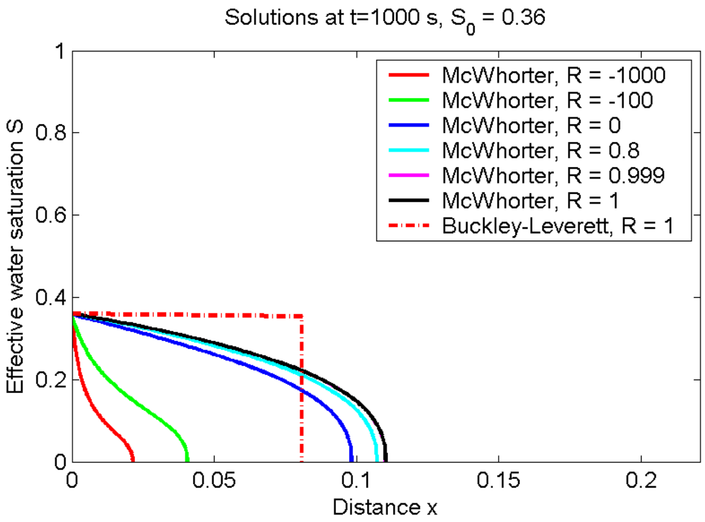
Solutions at t=1000 s, S0 = 0.33
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15



Solutions at t=1000 s, S_0 = 0.34
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2



Solutions at t=1000 s, S_0 = 0.35
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2



Solutions at t=1000 s, S_0 = 0.36
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
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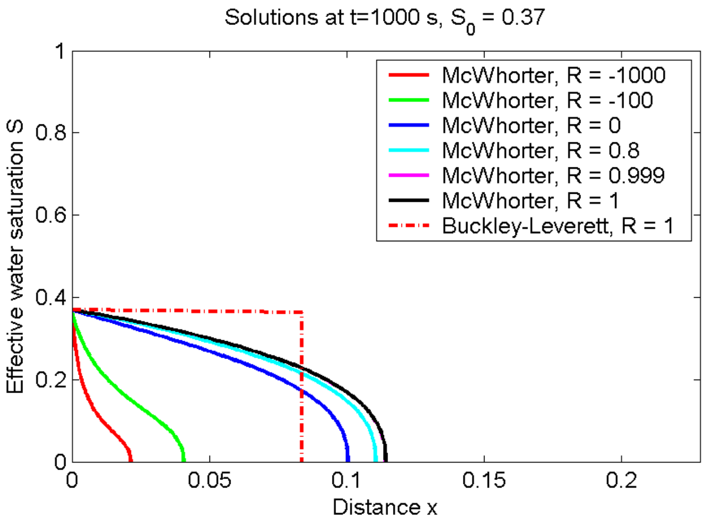
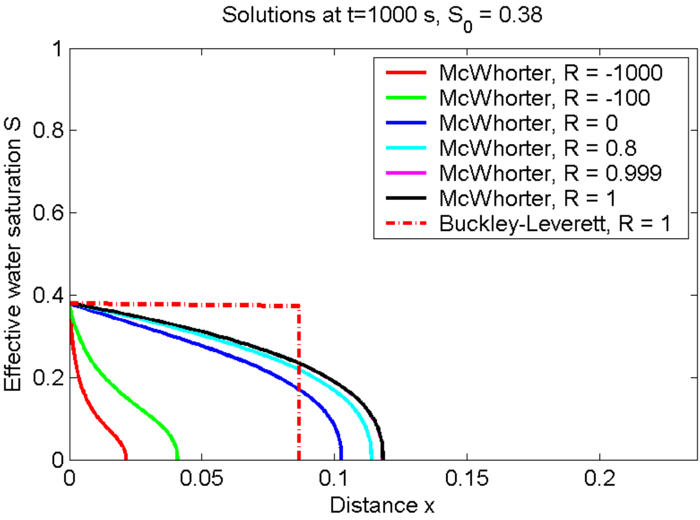
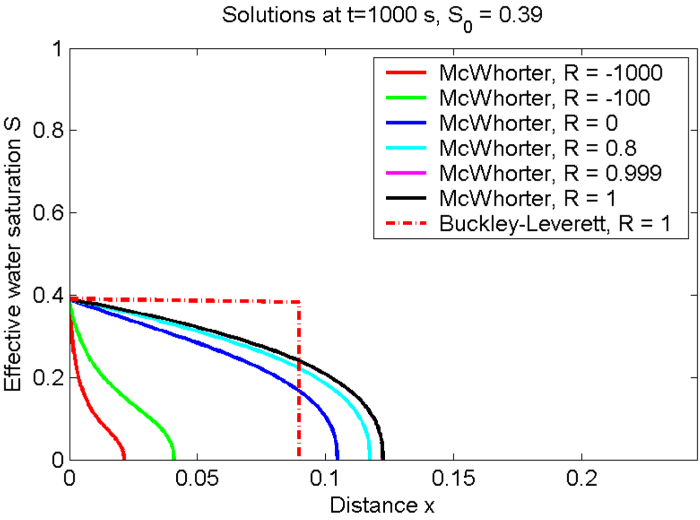


Solutions at t=1000 s, S_0 = 0.37
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

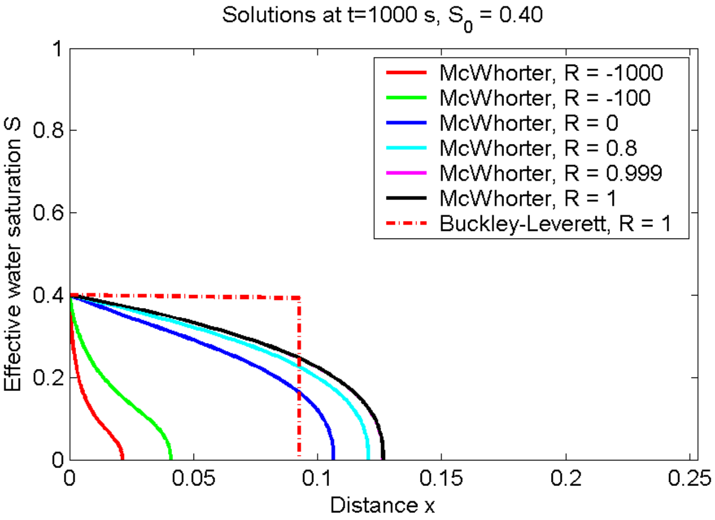


Solutions at t=1000 s, S0 = 0.38
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2


Solutions at t=1000 s, S_0 = 0.39
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2

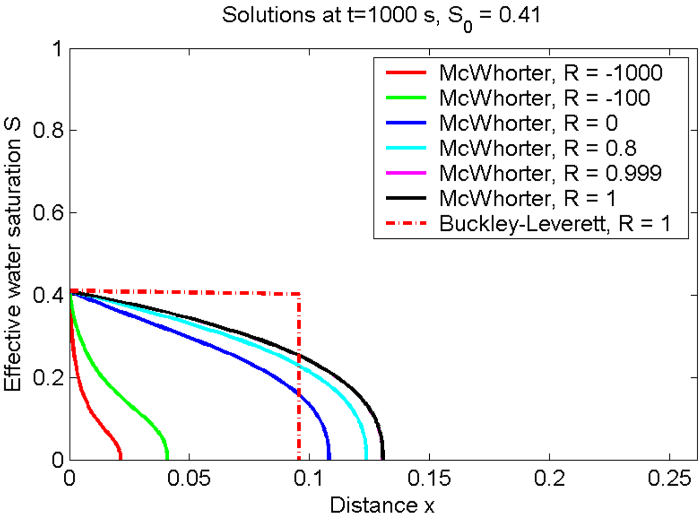
Troubles and Questions

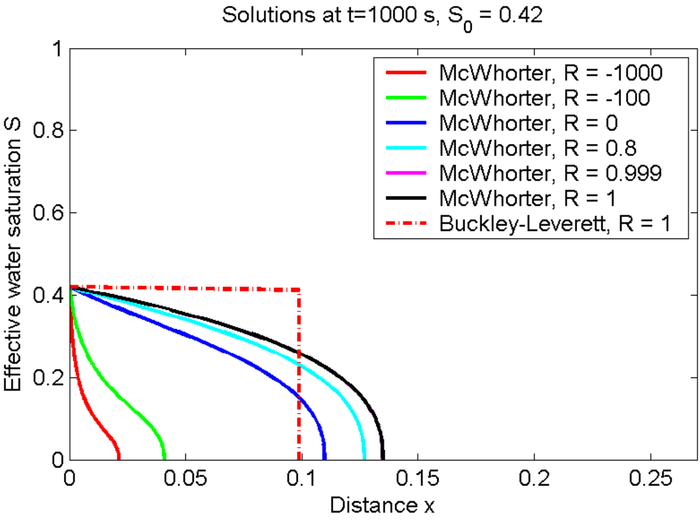
$S_0 \longrightarrow 1$ Demonstration



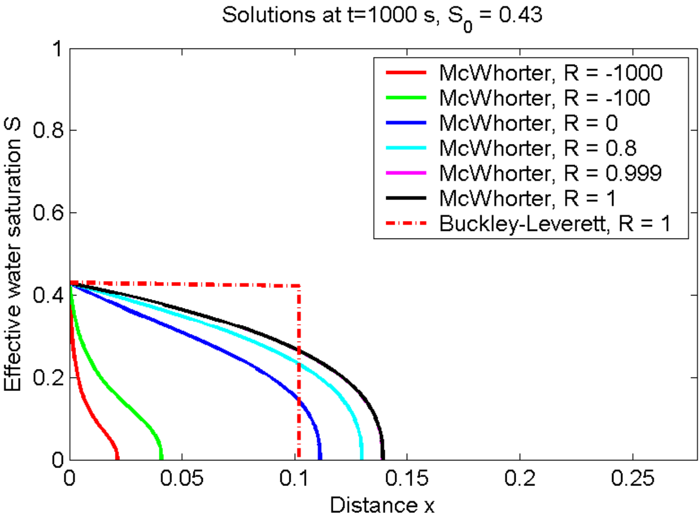
Troubles and Questions

$S_0 \rightarrow 1$ Demonstration

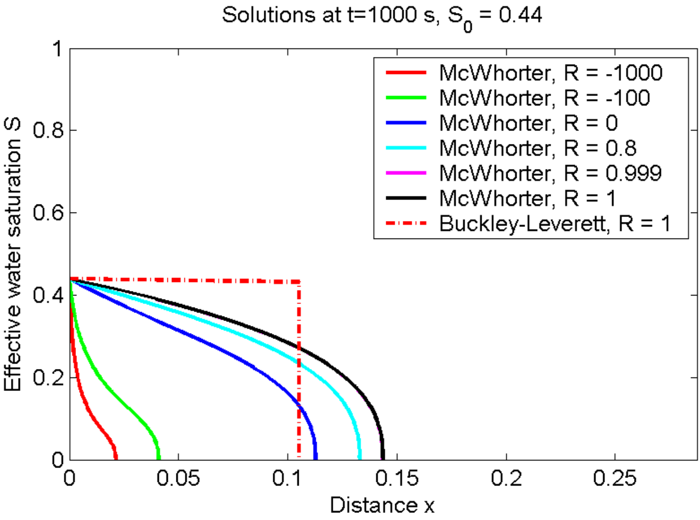




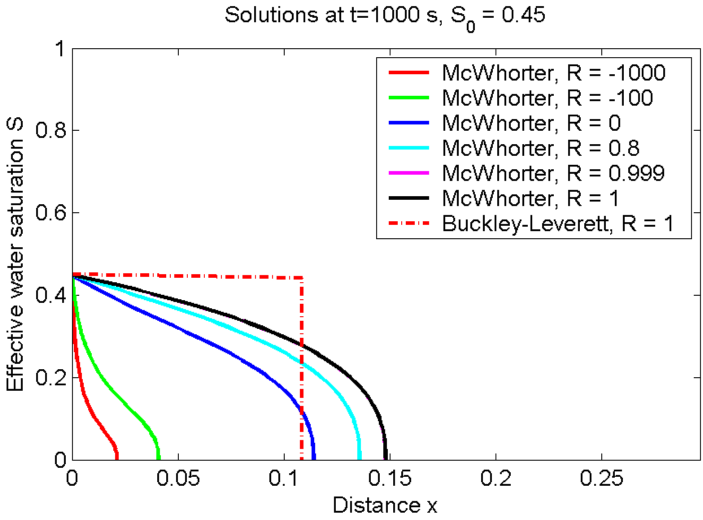
Solutions at t=1000 s, S0 = 0.42
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
0.25
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



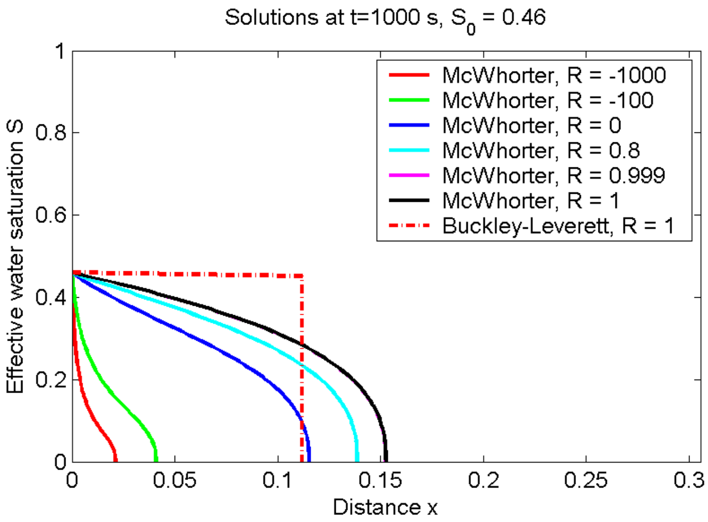
Solutions at t=1000 s, S0 = 0.43
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



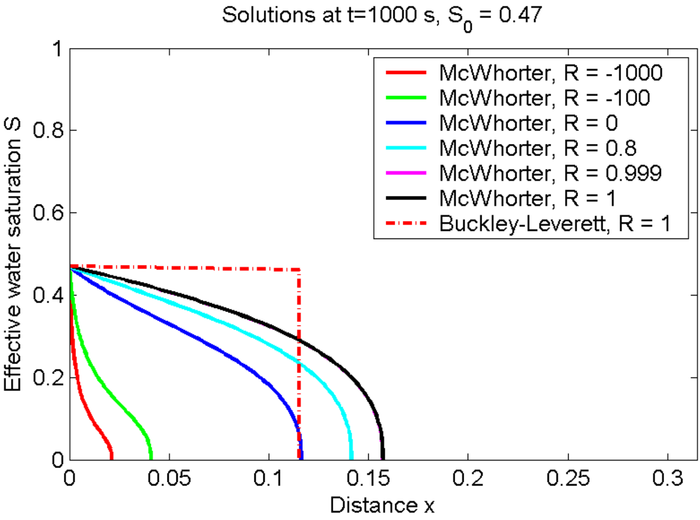
Solutions at t=1000 s, S0 = 0.44
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2
0.25



Solutions at t=1000 s, S_0 = 0.45
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
0.25
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S0 = 0.46
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2
0.25
0.3

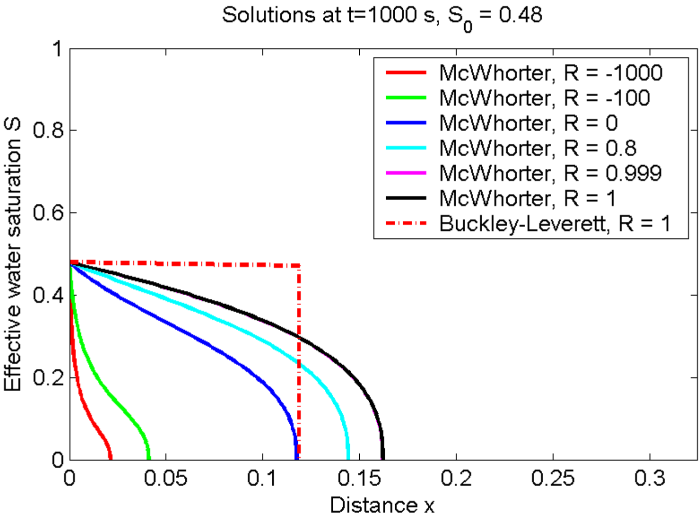


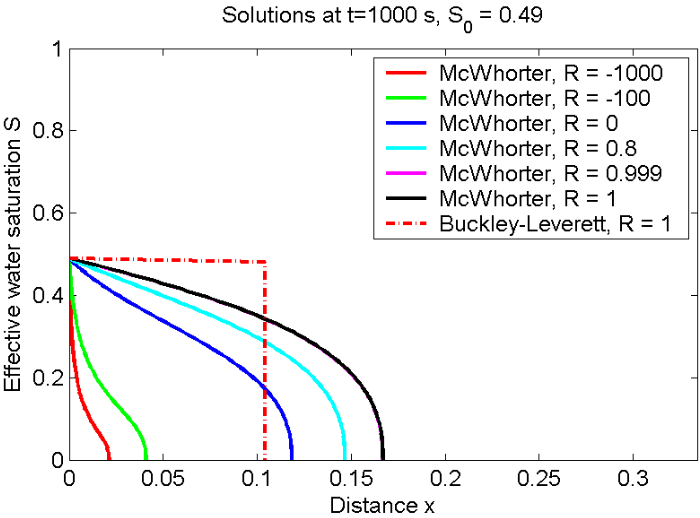
Solutions at t=1000 s, S_0 = 0.47
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
0.3
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

Troubles and Questions

$S_0 \to 1$ Demonstration

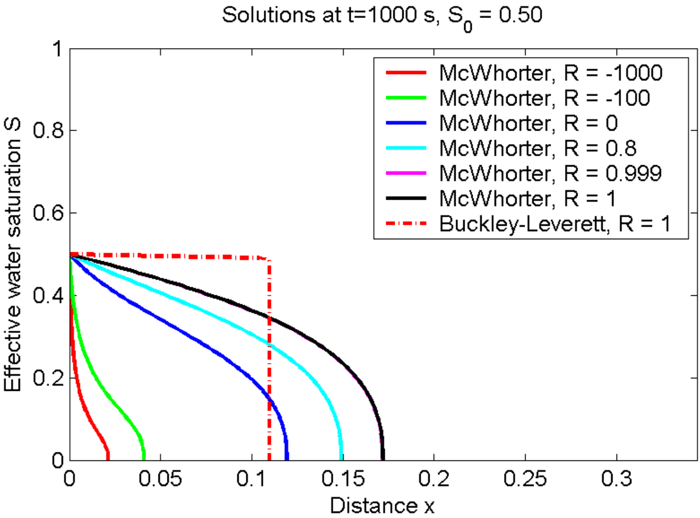
Mathematical Modelling Group

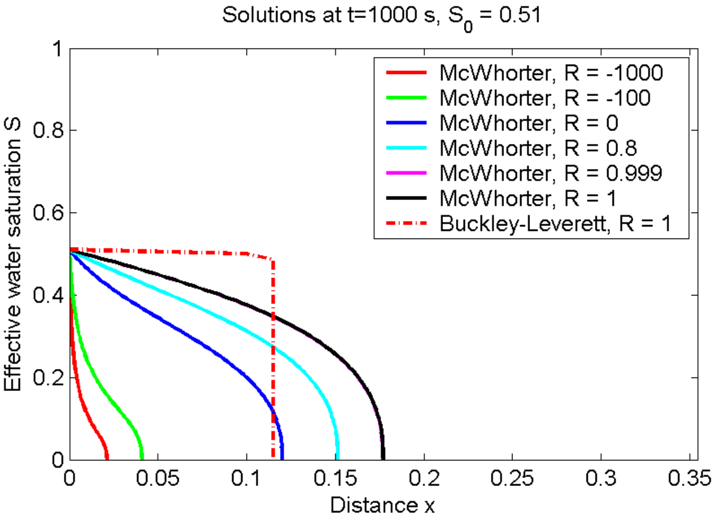



Solutions at t=1000 s, S_0 = 0.49
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
0.3

Troubles and Questions

$S_0 \to 1$ Demonstration

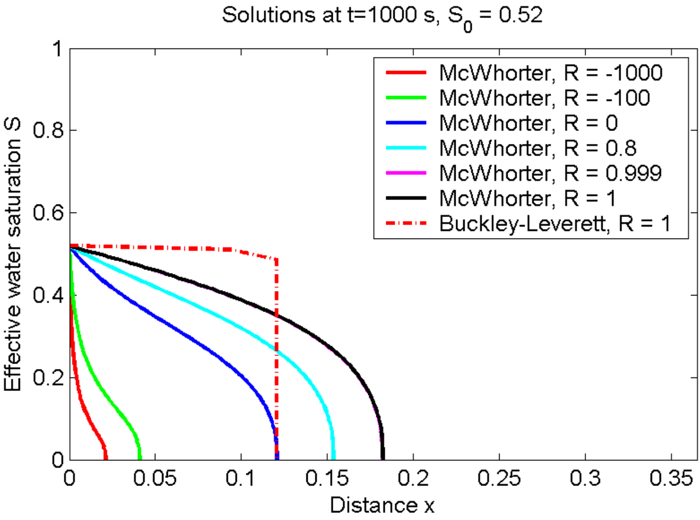




Solutions at t=1000 s, S0 = 0.51
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35

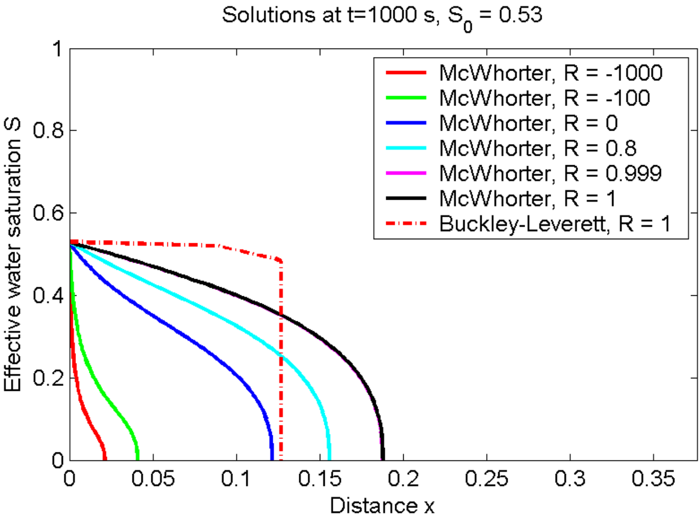
Troubles and Questions

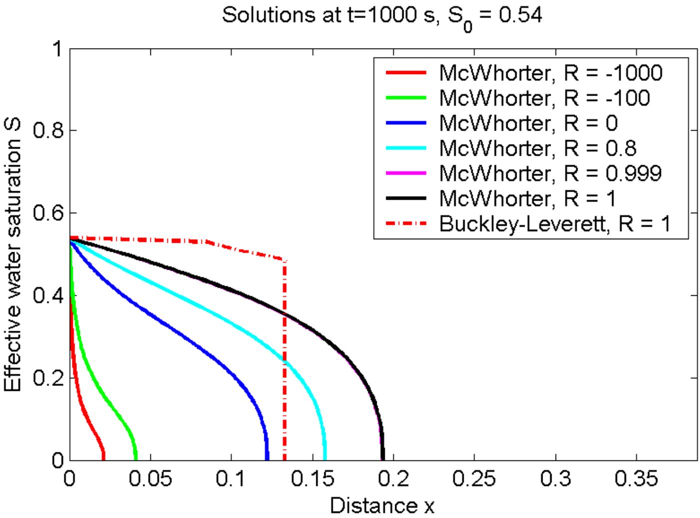
$S_0 \rightarrow 1$ Demonstration



Troubles and Questions

$S_0 \longrightarrow 1$ Demonstration

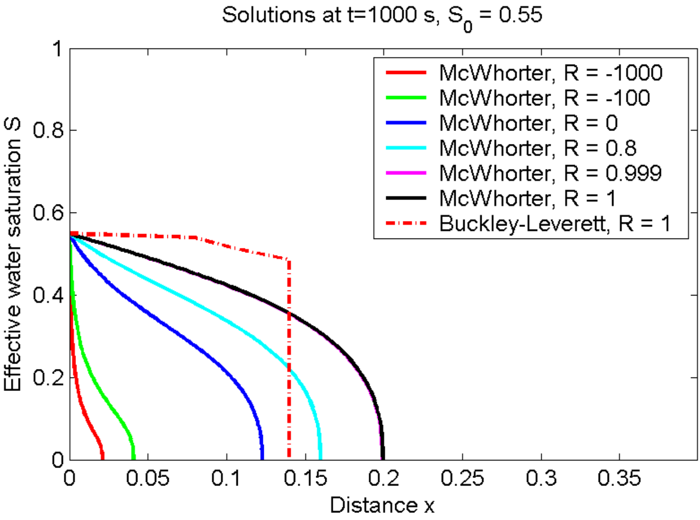


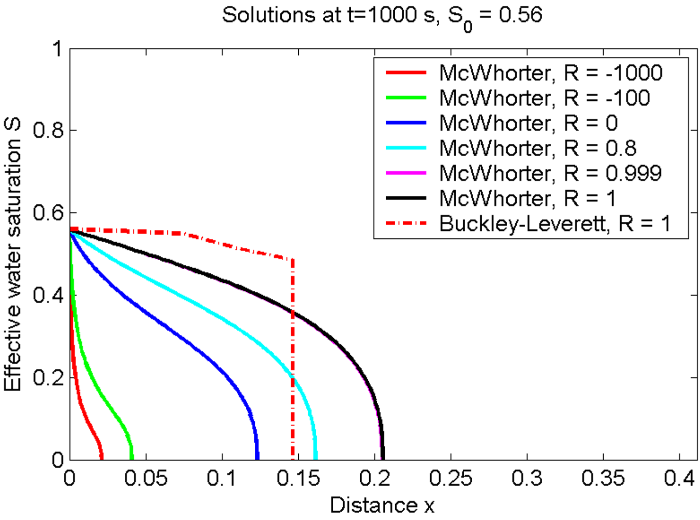


Solutions at t=1000 s, S0 = 0.54
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

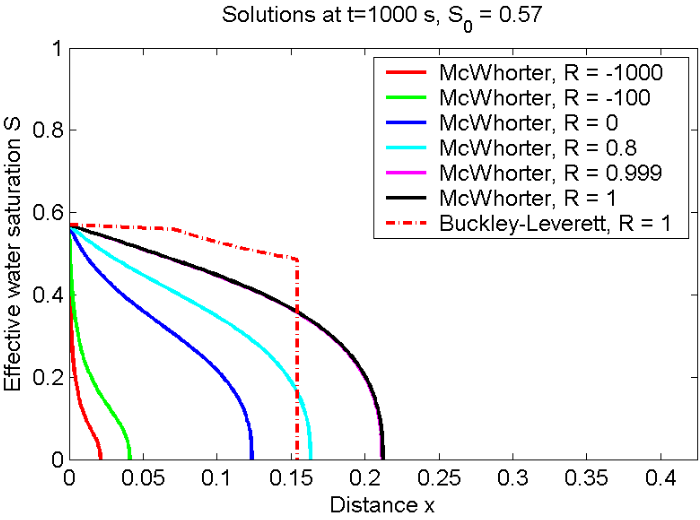
Troubles and Questions

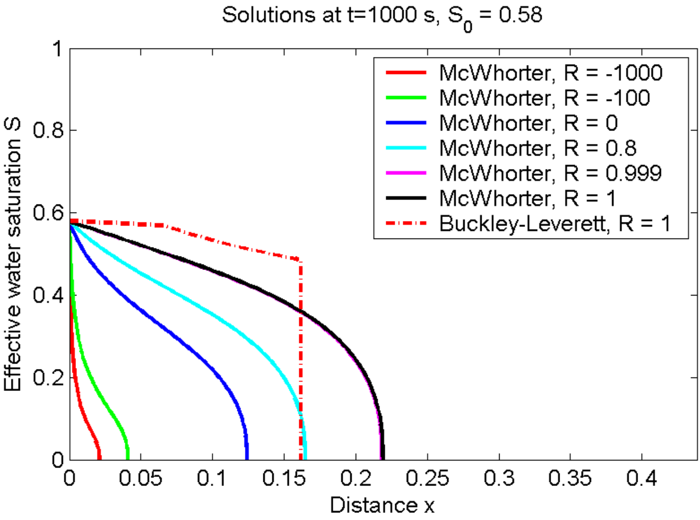
$S_0 \rightarrow 1$ Demonstration

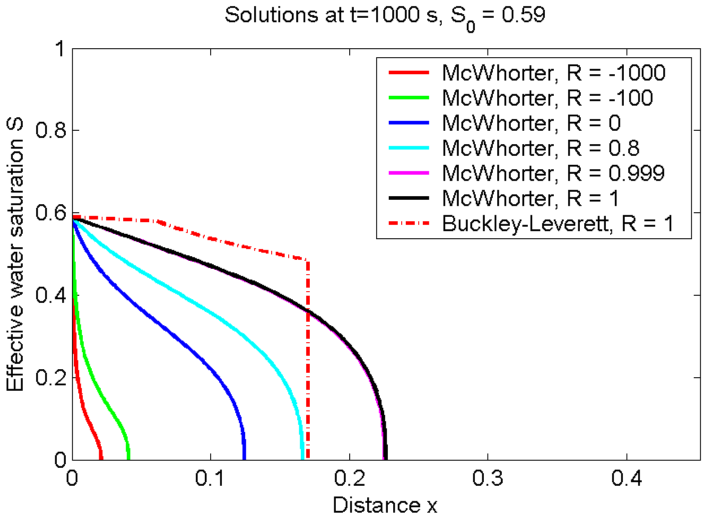


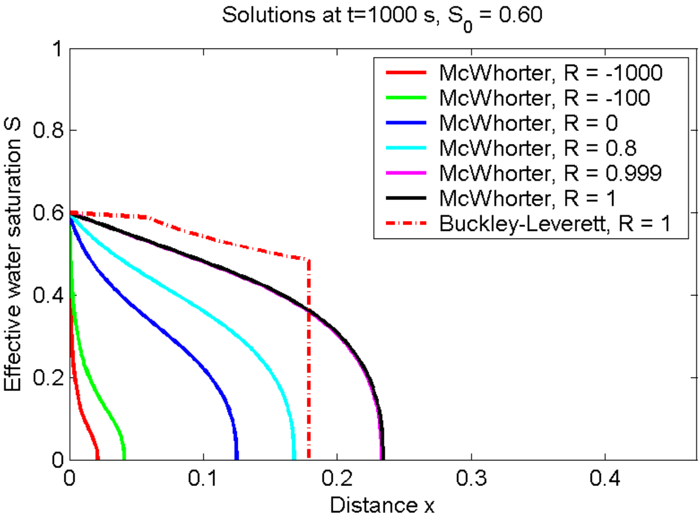


Solutions at t=1000 s, S0 = 0.56
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

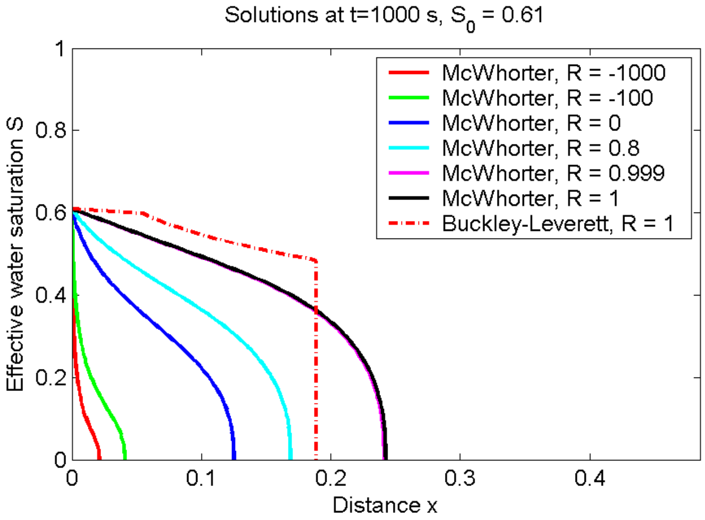

Solutions at t=1000 s, S_0 = 0.57
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4


Solutions at t=1000 s, S_0 = 0.58
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4

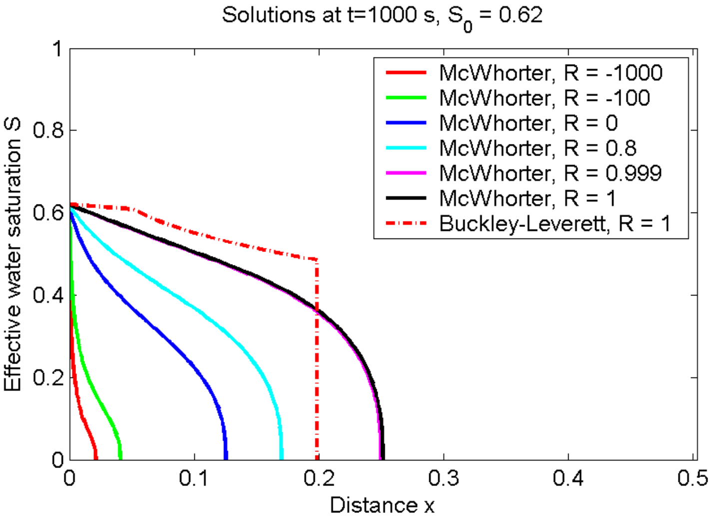

Solutions at t=1000 s, S_0 = 0.59
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S0 = 0.60
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0.1
0.2
0.3
0.4



Solutions at t=1000 s, S0 = 0.61
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4

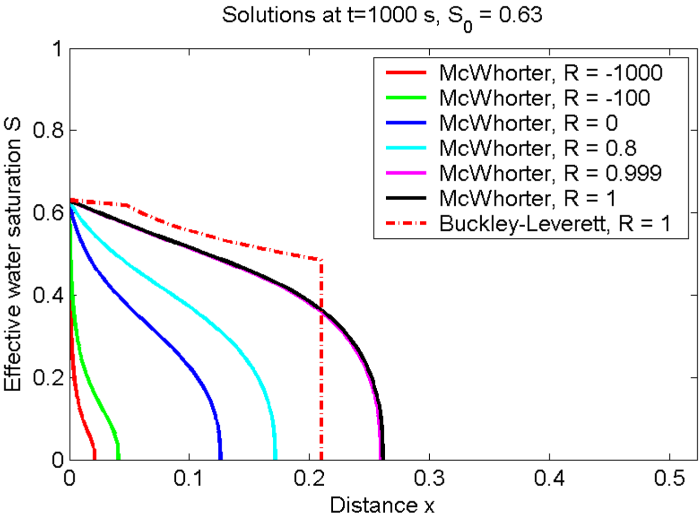


Solutions at t=1000 s, S0 = 0.62
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

Troubles and Questions

$S_0 \rightarrow 1$ Demonstration

Mathematical Modelling Group



Benchmark Solutions for Two-Phase Porous-Media Flow

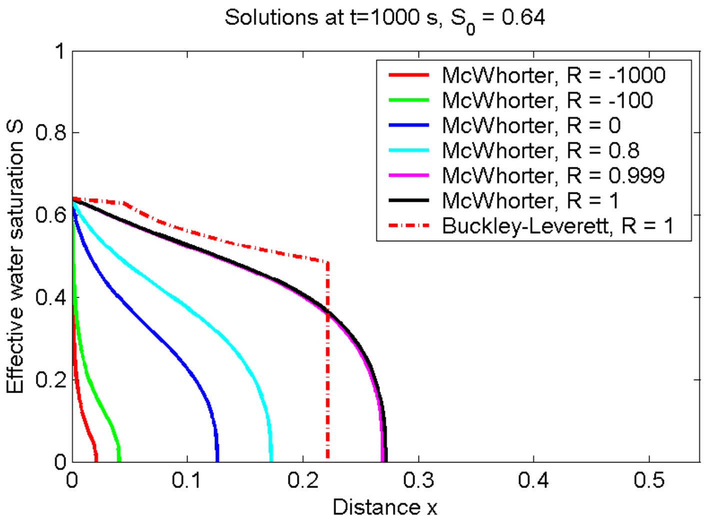
Radek Fučík

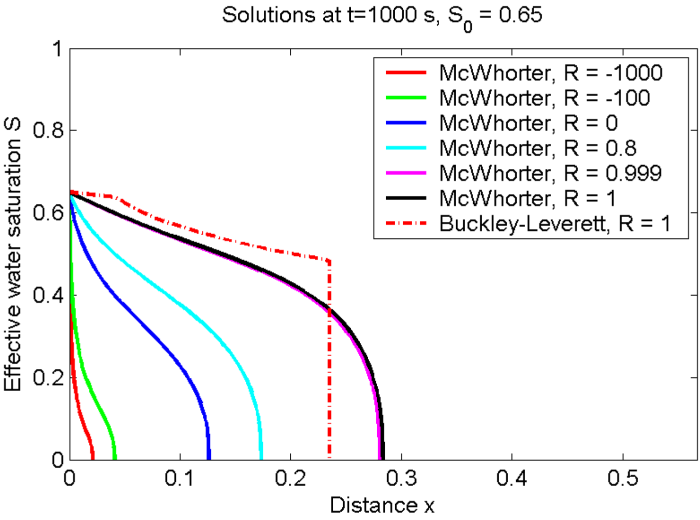
Introduction
Motivation
Mathematical model

Exact Solution
McWhorter and Sunada Problem
Integral equation

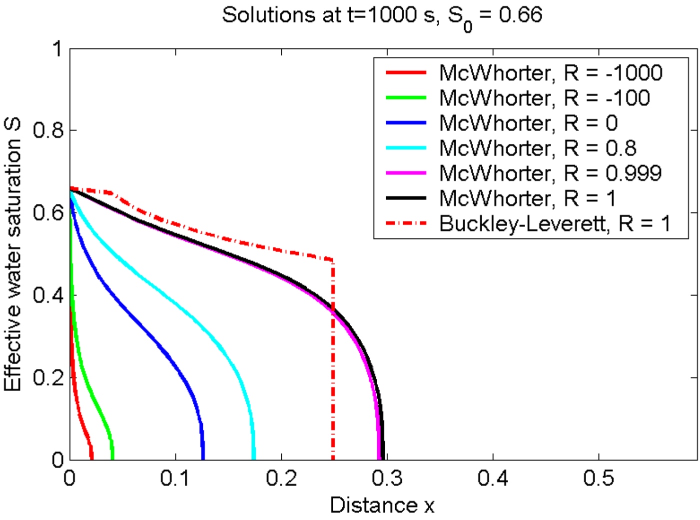
Troubles and Questions
First Iteration Analysis
Modified Integral Equation
Generalizations
$S_0 \longrightarrow 1$ Demonstration
Finiteness of $A(S_0)$ as $S_0 \longrightarrow 1$
Heterogeneous Porous Media

Conclusion

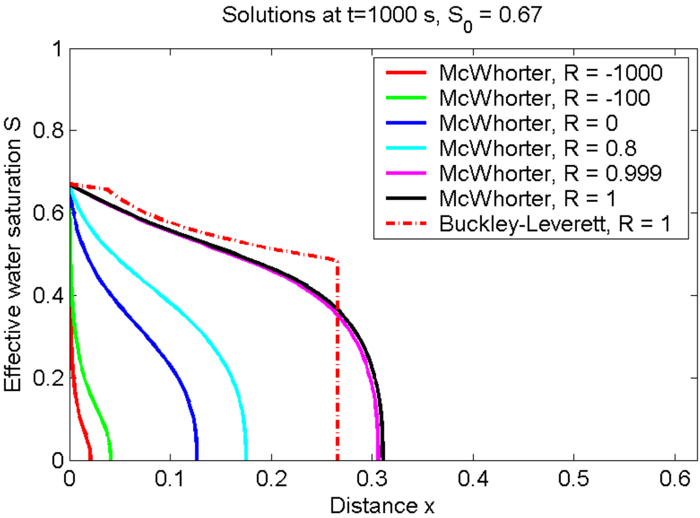


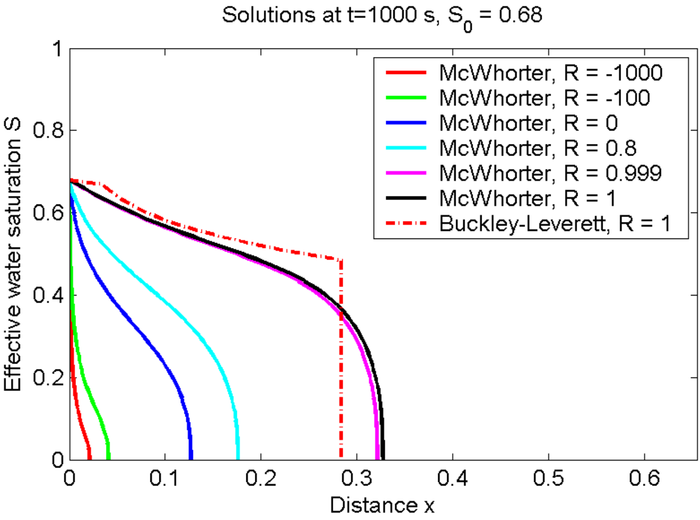


Solutions at t=1000 s, S_0 = 0.65
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5

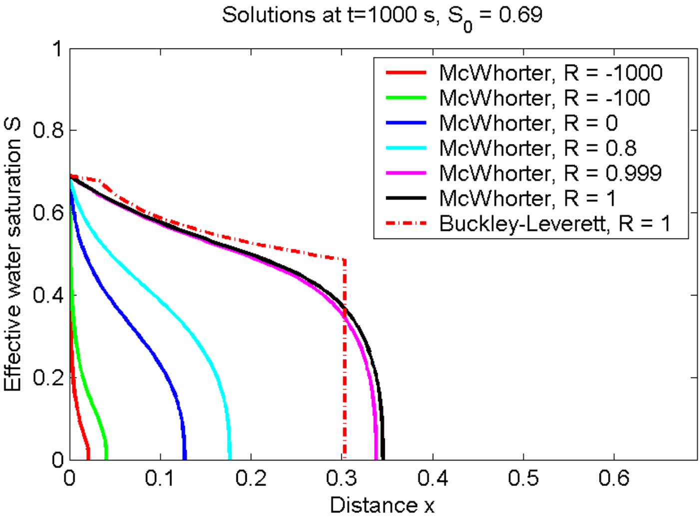


Solutions at t=1000 s, S0 = 0.66
Effective water saturation S
Distance x
0
0.1
0.2
0.3
0.4
0.5
0.2
0.4
0.6
0.8
1
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

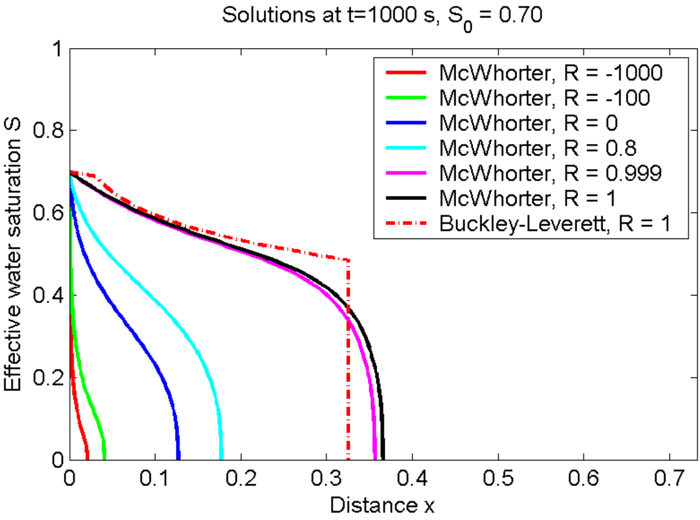

Solutions at t=1000 s, S0 = 0.67
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6



Solutions at t=1000 s, S0 = 0.68
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



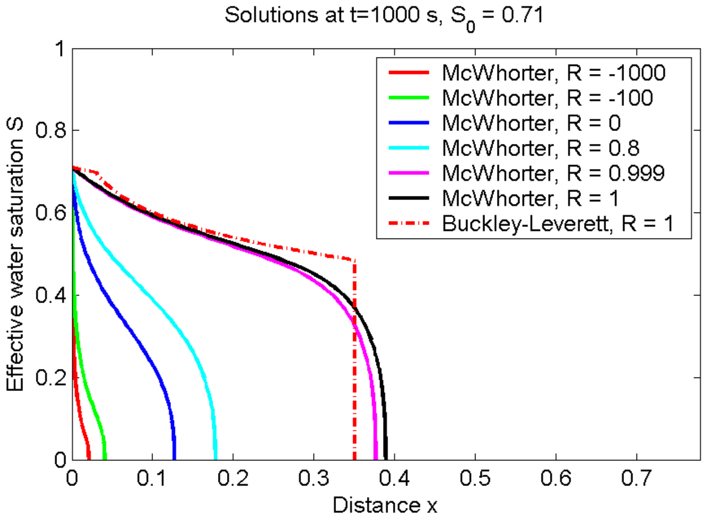
Solutions at t=1000 s, S_0 = 0.69
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.1
0.2
0.3
0.4
0.5
0.6
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S_0 = 0.70
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7

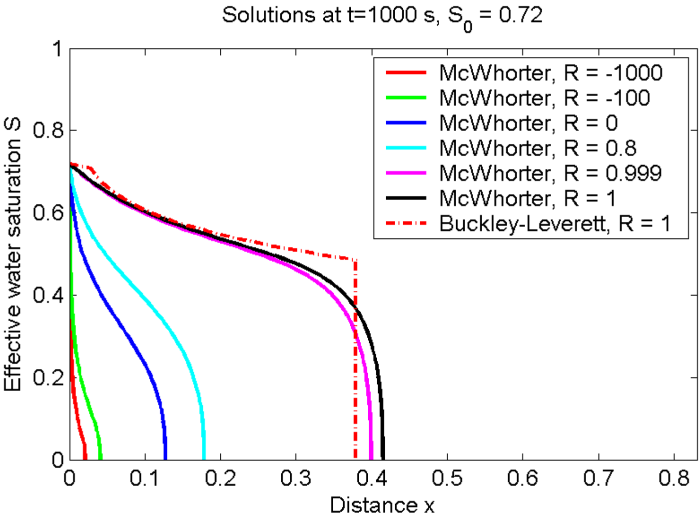
Troubles and Questions

$S_0 \to 1$ Demonstration



Troubles and Questions

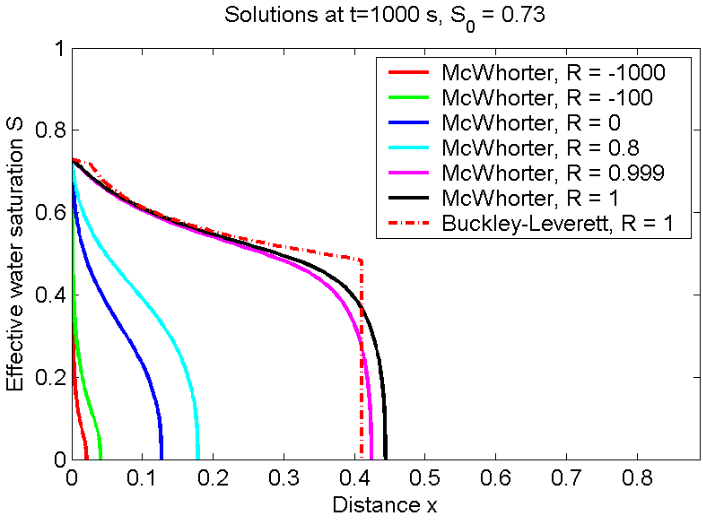
$S_0 \rightarrow 1$ Demonstration

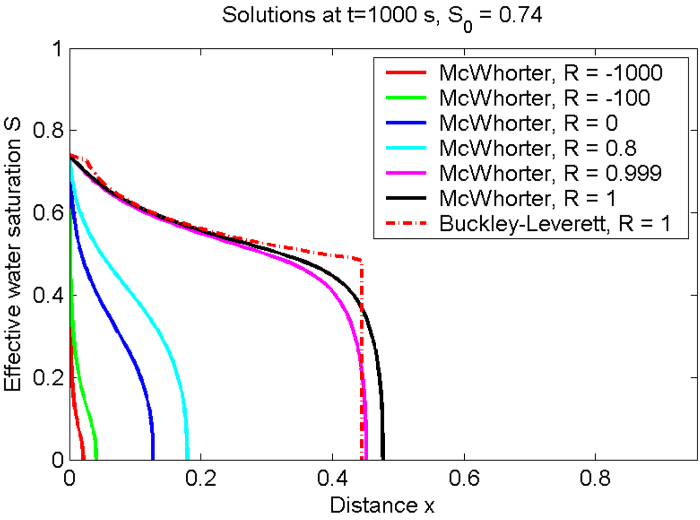


Troubles and Questions

$S_0 \rightarrow 1$ Demonstration

Mathematical Modelling Group

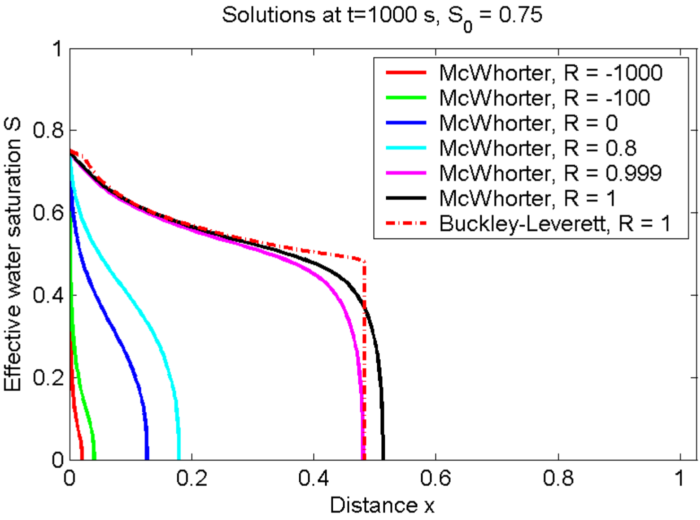




Solutions at t=1000 s, S0 = 0.74
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

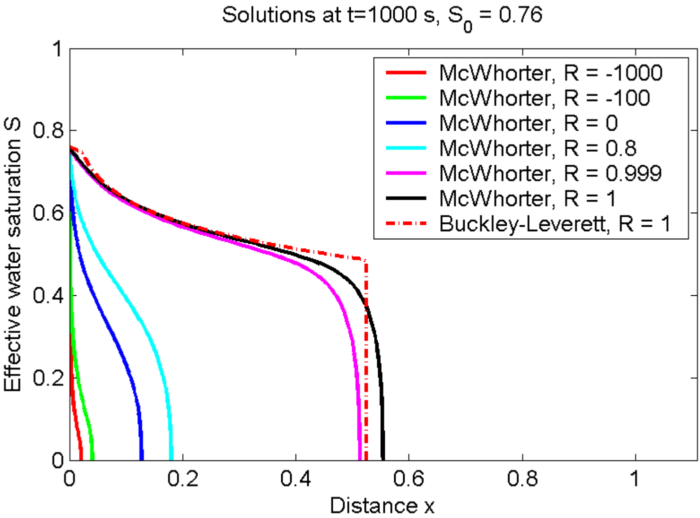
Troubles and Questions

$S_0 \to 1$ Demonstration



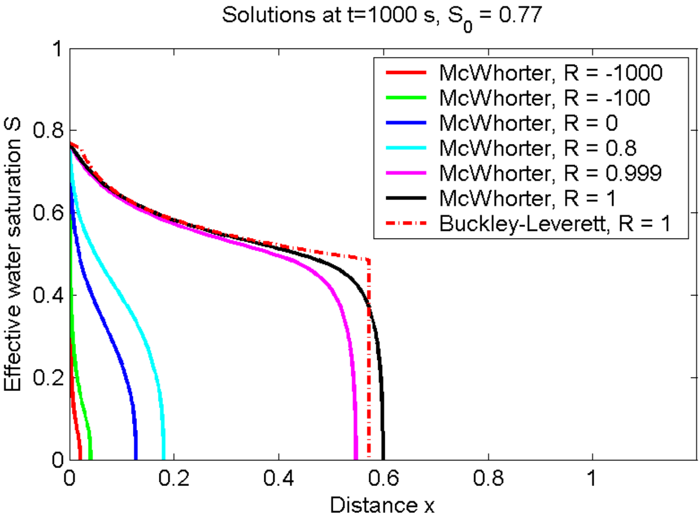
Troubles and Questions

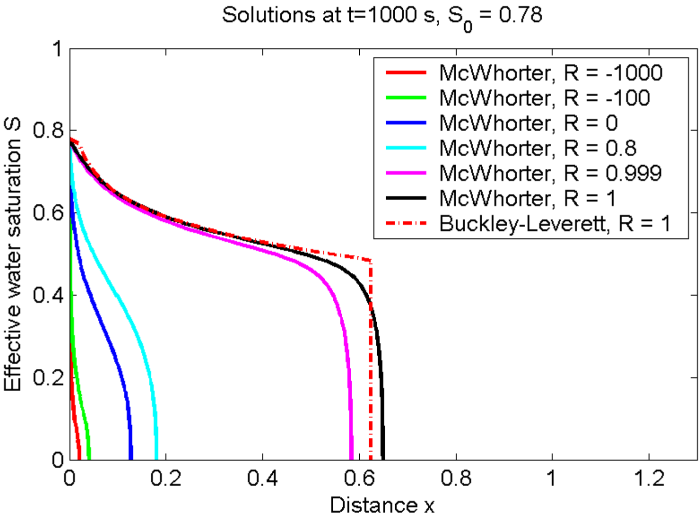
$S_0 \longrightarrow 1$ Demonstration



Troubles and Questions

$S_0 \to 1$ Demonstration

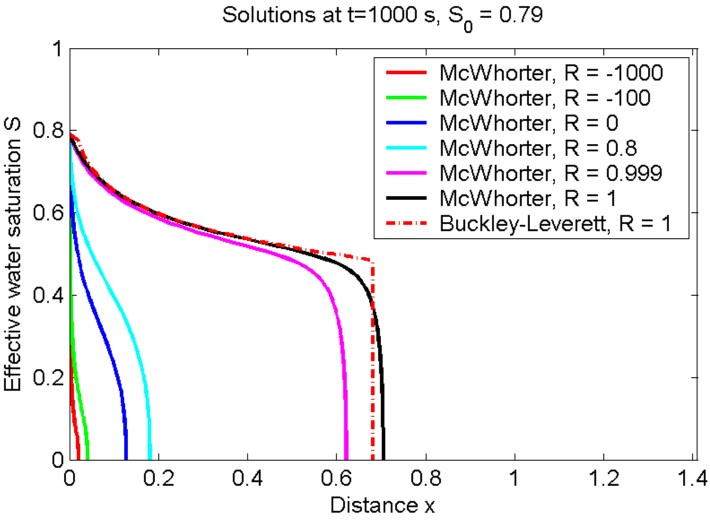


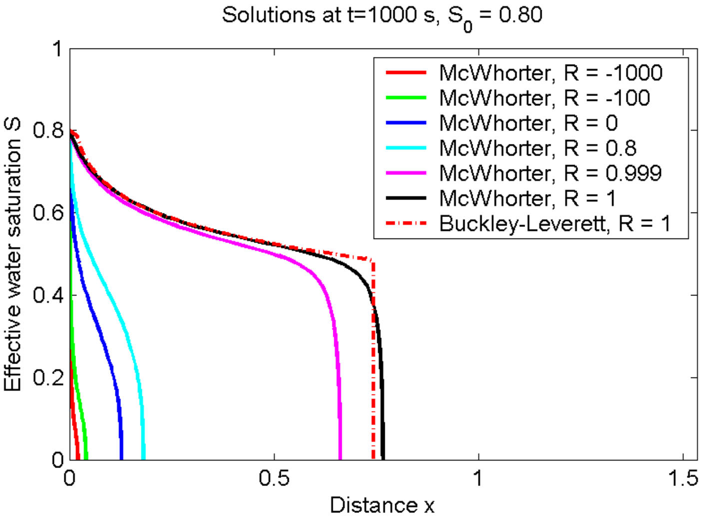


Solutions at t=1000 s, S0 = 0.78
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
1.2

Troubles and Questions

$S_0 \rightarrow 1$ Demonstration

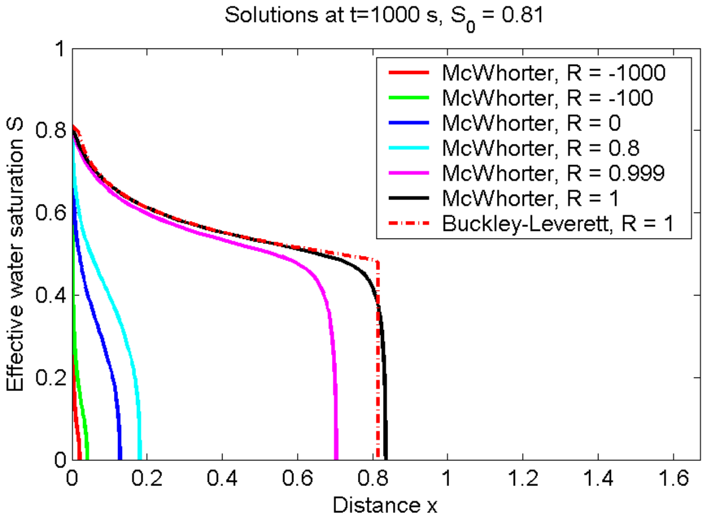


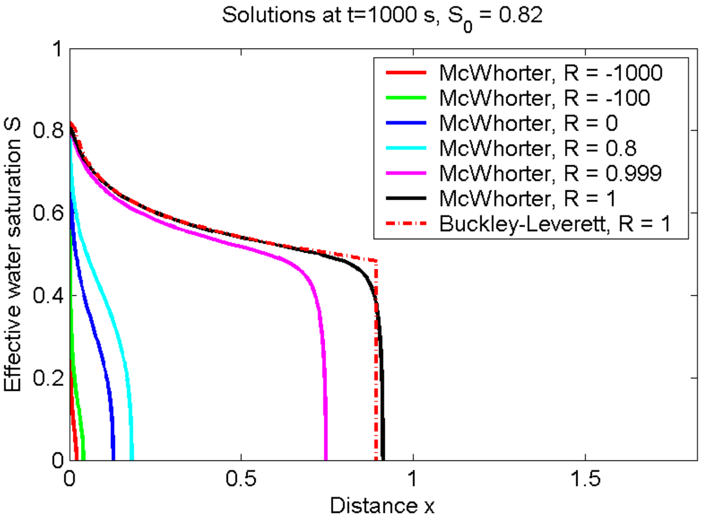


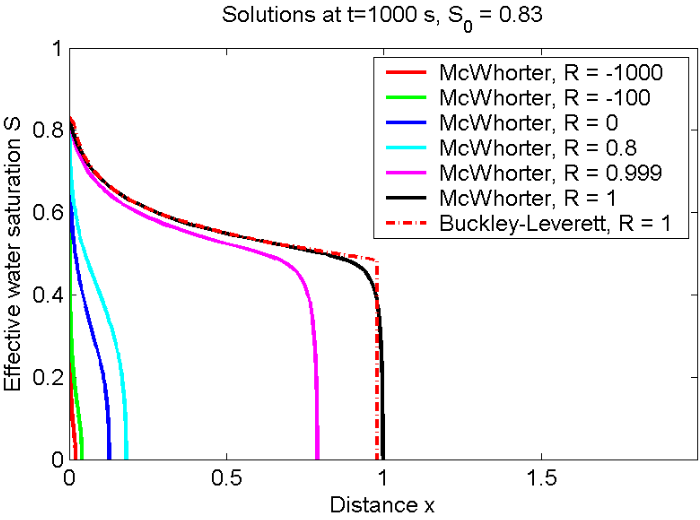
Solutions at t=1000 s, S0 = 0.80
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

Troubles and Questions

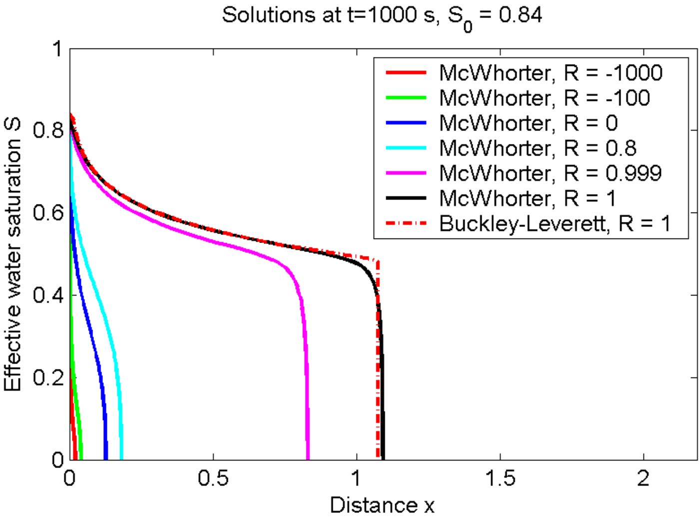
$S_0 \to 1$ Demonstration




Solutions at t=1000 s, S0 = 0.82
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5



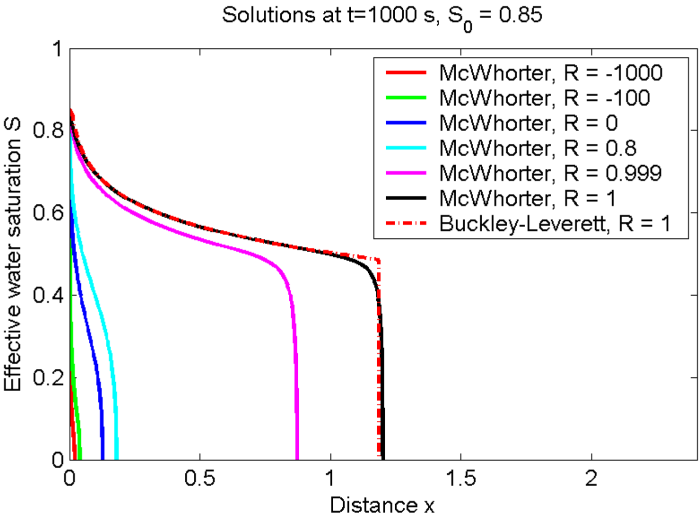
Solutions at t=1000 s, S0 = 0.83
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S0 = 0.84
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1

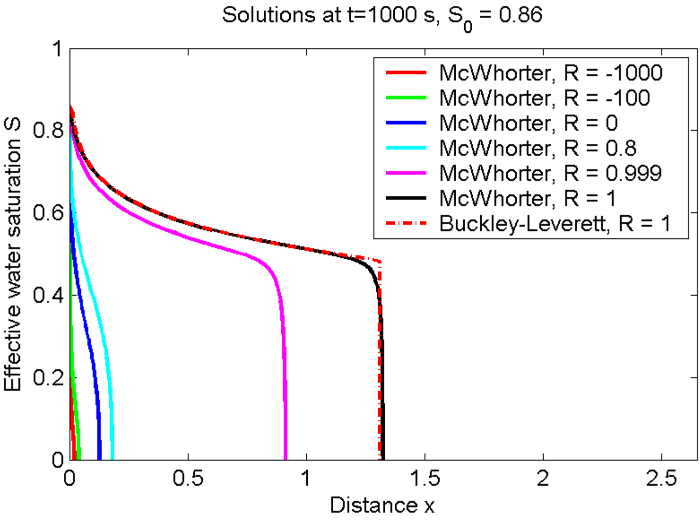
Troubles and Questions

$S_0 \to 1$ Demonstration



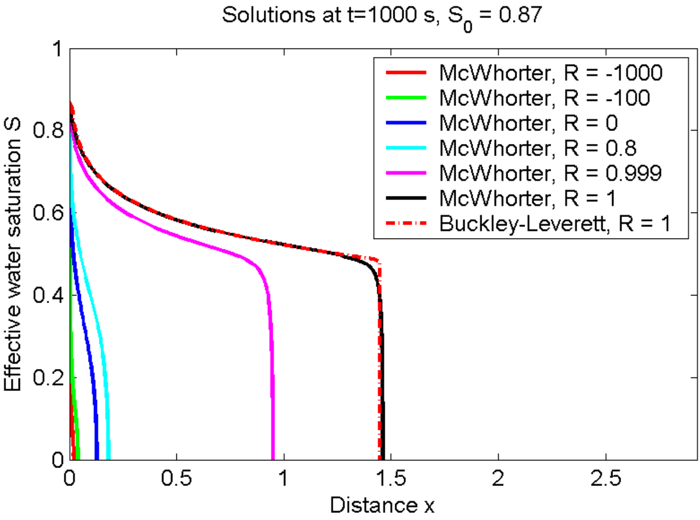
Troubles and Questions

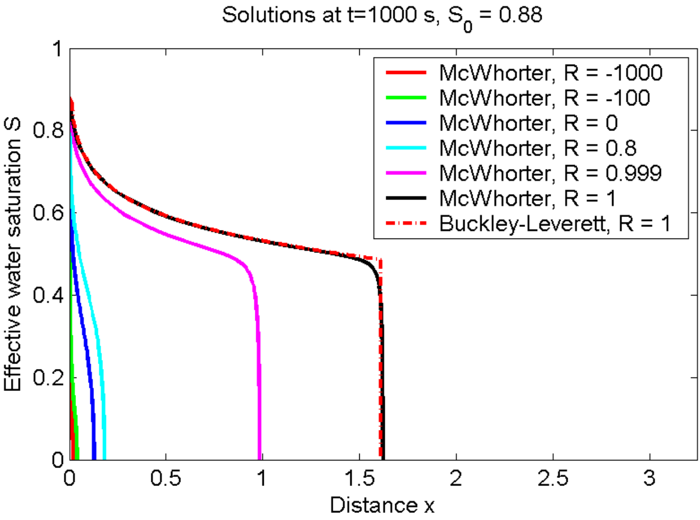
$S_0 \rightarrow 1$ Demonstration

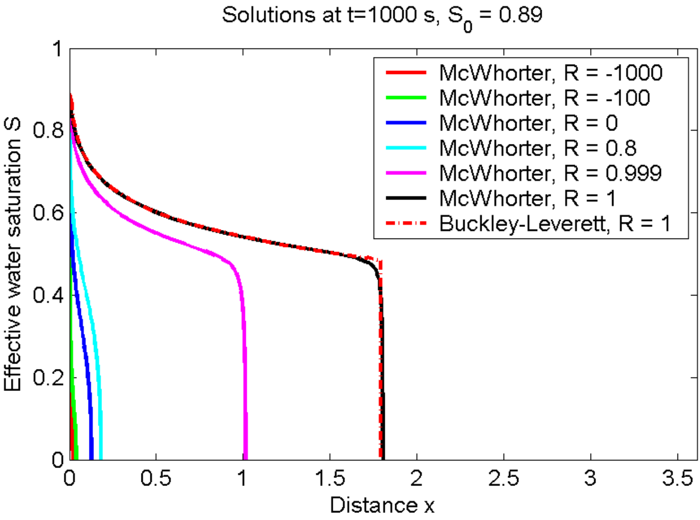


Troubles and Questions

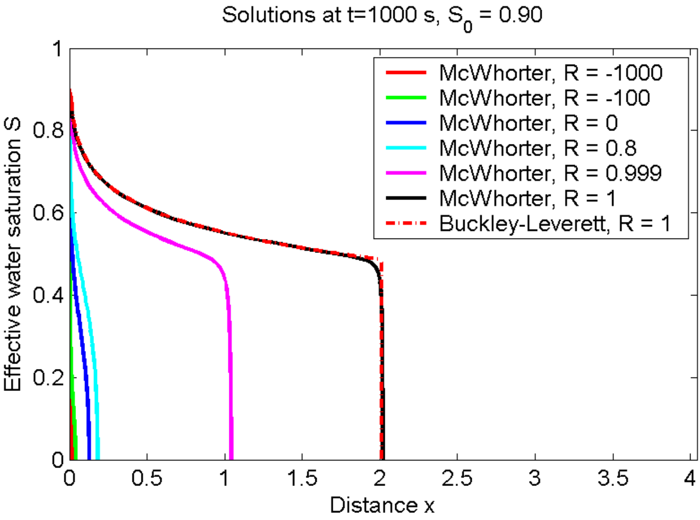
$S_0 \to 1$ Demonstration



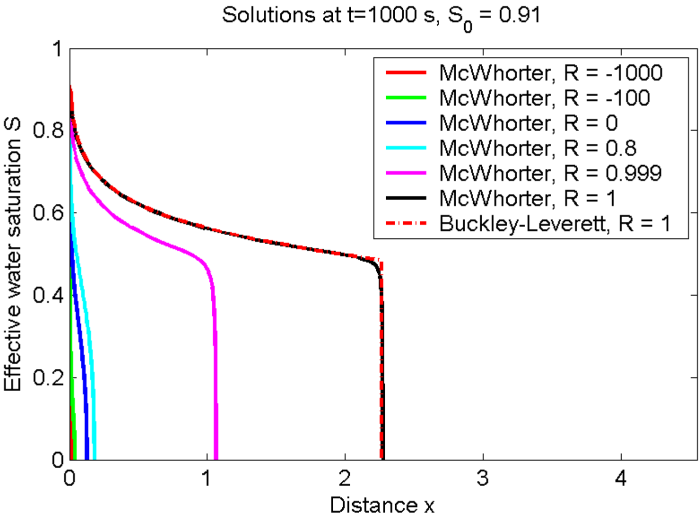

Solutions at t=1000 s, S_0 = 0.88
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2
2.5
3



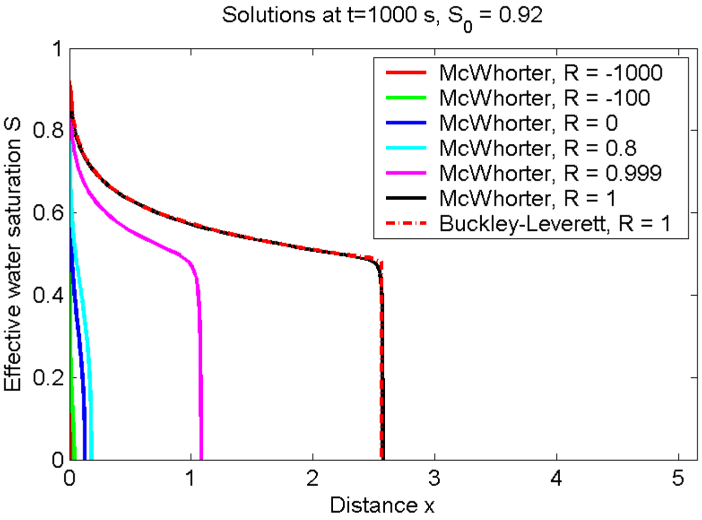
Solutions at t=1000 s, S_0 = 0.89
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3
3.5



Solutions at t=1000 s, S0 = 0.90
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
0.5
1
1.5
2
2.5
3
3.5
4
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



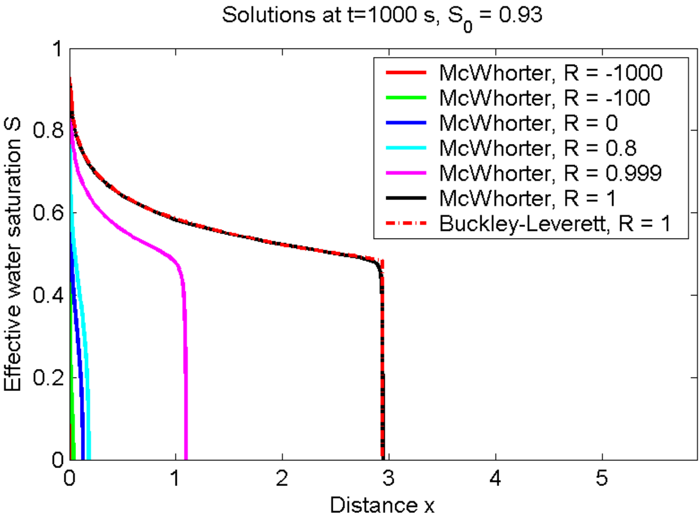
Solutions at t=1000 s, S_0 = 0.91
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4

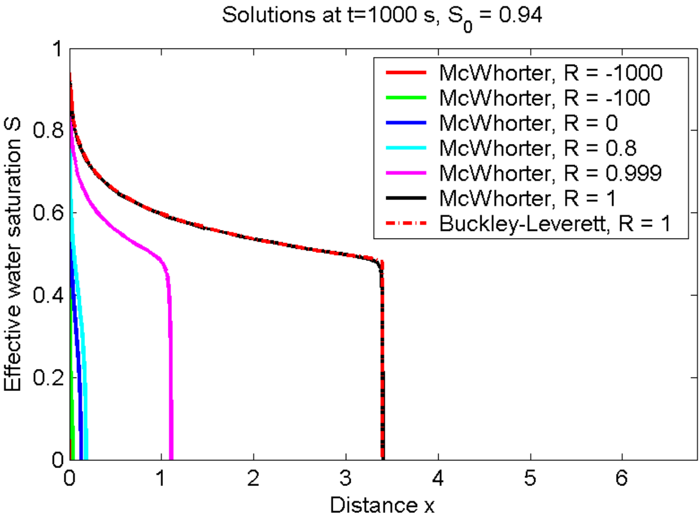


Solutions at t=1000 s, S_0 = 0.92
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5

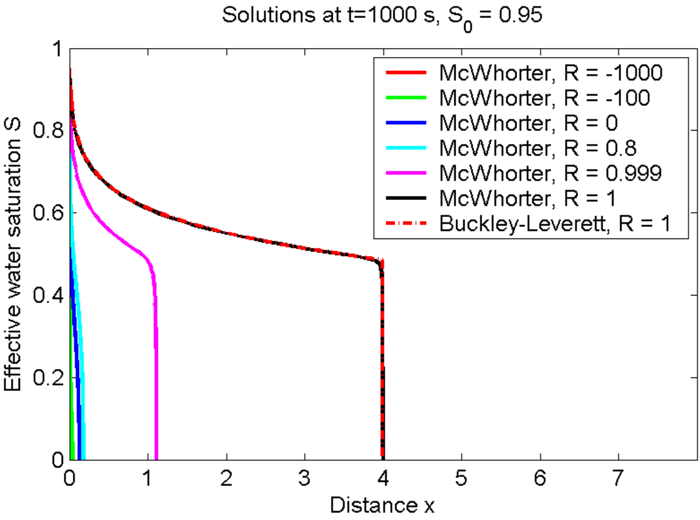
Troubles and Questions

$S_0 \to 1$ Demonstration

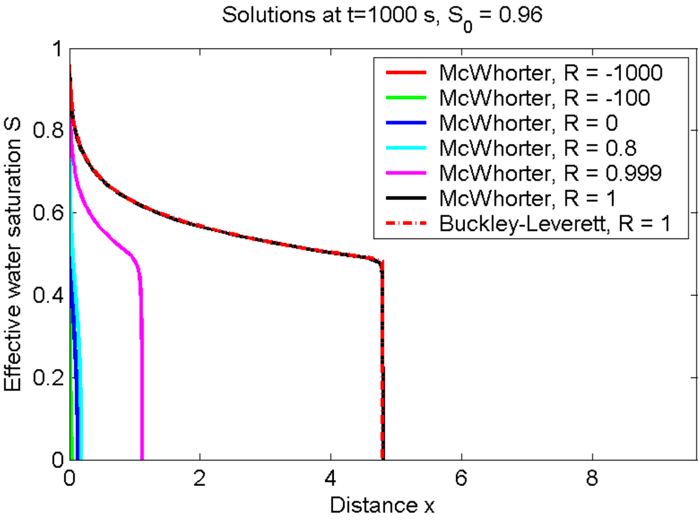




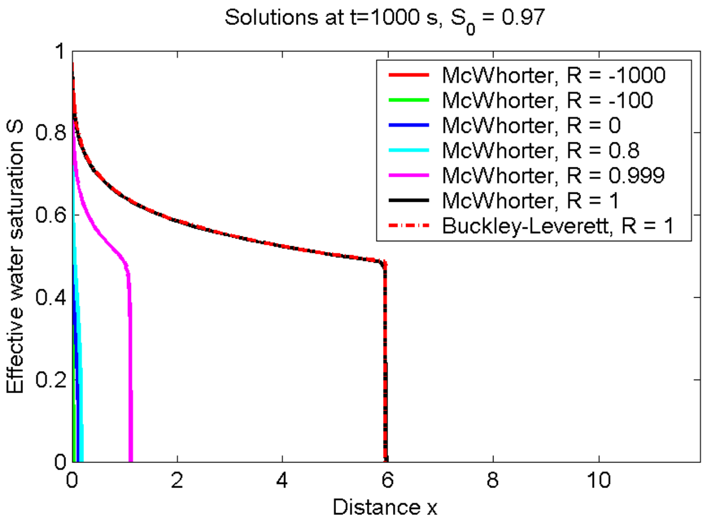
Solutions at t=1000 s, S0 = 0.94
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5
6
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



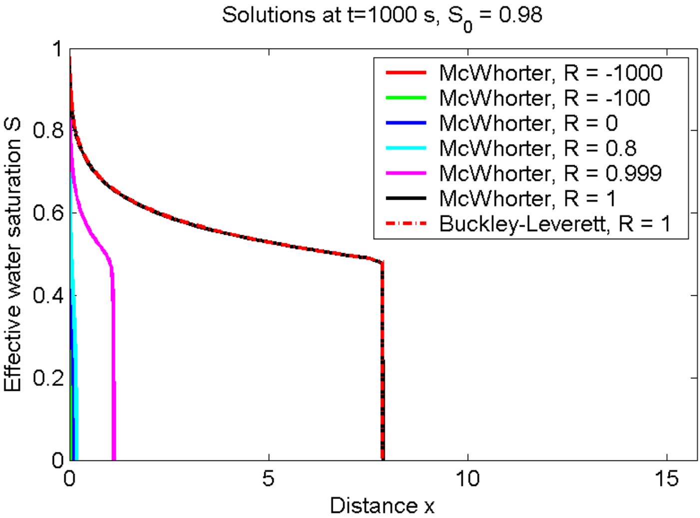
Solutions at t=1000 s, S_0 = 0.95
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5
6
7



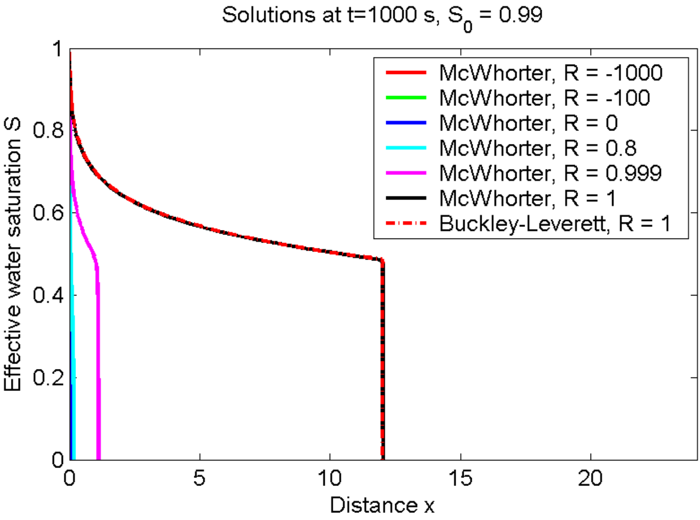
Solutions at t=1000 s, S0 = 0.96
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8



Solutions at t=1000 s, S0 = 0.97
Effective water saturation S
Distance x
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1



Solutions at t=1000 s, S_0 = 0.98
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
Effective water saturation S
Distance x
0
0.2
0.4
0.6
0.8
1
0
5
10
15



Solutions at t=1000 s, S_0 = 0.99
Effective water saturation S
Distance x
McWhorter, R = -1000
McWhorter, R = -100
McWhorter, R = 0
McWhorter, R = 0.8
McWhorter, R = 0.999
McWhorter, R = 1
Buckley-Leverett, R = 1
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20

Troubles and Questions

Overview

1. The suggested iterative scheme fails for $R = 1$ and $S_0 > S_{0\ crit}$ due to singularity in the first iteration !
2. Is there another way to obtain F ? Yes.
3. Is it possible to generalize the range of the input parameters ? Yes. $R \in (-\infty, 1]$
4. Is the Buckley-Leverett solution the limiting solution as $S_0 \to 1$?

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Benchmark Solutions for Two-Phase Porous-Media Flow

Radek Fučík

Introduction
Motivation
Mathematical model

Exact Solution
McWhorter and Sunada Problem
Integral equation

Troubles and Questions
First Iteration Analysis
Modified Integral Equation
Generalizations
$S_0 \longrightarrow 1$ Demonstration
Finiteness of $A(S_0)$ as $S_0 \longrightarrow 1$
Heterogeneous Porous Media

Conclusion

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Troubles and Questions

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Mathematical Modelling Group

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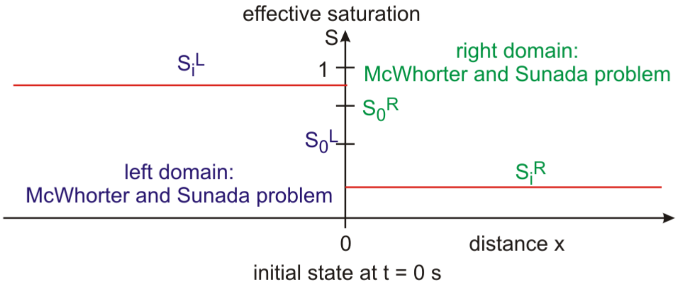
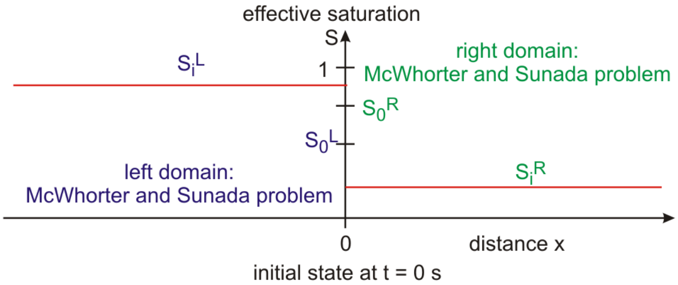
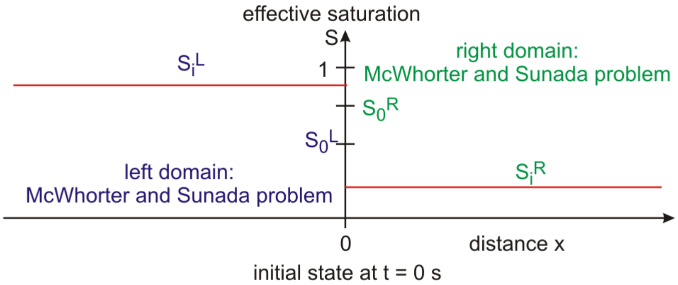
6. Is it possible to derive exact solution for the heterogeneous porous media ?

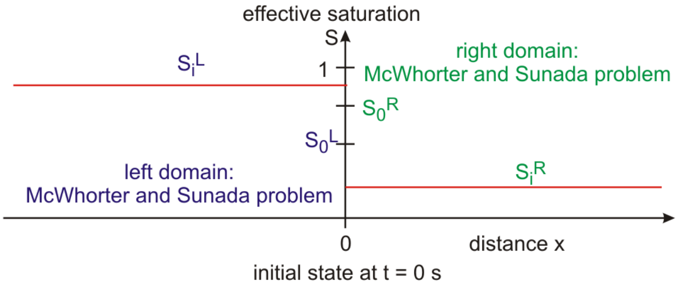

effective saturation
S
1
S_i^L
right domain:
McWhorter and Sunada problem
S_0^R
S_0^L
left domain:
McWhorter and Sunada problem
S_i^R
0
distance x
initial state at t = 0 s



- Combination of two exact solutions for the homogeneous media problems

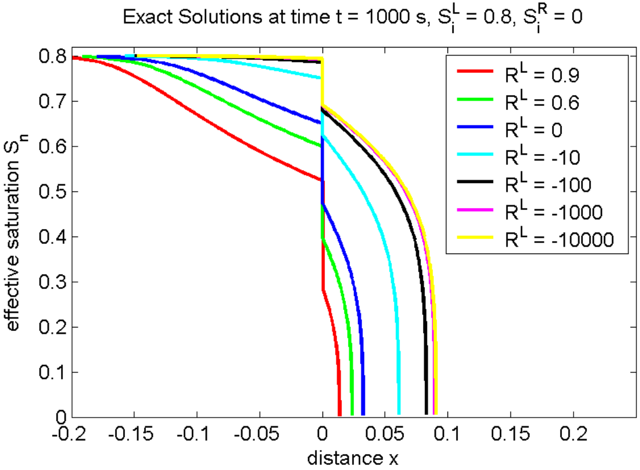


- Combination of two exact solutions for the homogeneous media problems
- Interfacial conditions:



- Combination of two exact solutions for the homogeneous media problems
- Interfacial conditions:
  - $A^L R^L = A^R R^R$
  - $R^R - R^R R^L + R^L = 0$
  - $S_0^R = \kappa(S_0^L)$

Example


Exact Solutions at time t = 1000 s, S_i^L = 0.8, S_i^R = 0
effective saturation S_n
distance x
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
R^L = 0.9
R^L = 0.6
R^L = 0
R^L = -10
R^L = -100
R^L = -1000
R^L = -10000

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4. Is the Buckley-Leverett solution the limiting solution as $S_0 \rightarrow 1$? Yes.
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Conclusion

Mathematical Modelling Group

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Thank you for your attention.